Exam I Answers.

1. A) The second Taylor polynomial $T_2(x)$ of $f(x)$ centered at 1 is

$$T_2(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2} (x - 1)^2$$

$$= e + e(x - 1) + \frac{e}{2} (x - 1)^2.$$

After simplification, $T_2(x) = \frac{e}{2} (x^2 + 1)$.

1. B) $f(x) - T_2(x) = R_2(x) = \frac{f^{(3)}(c)}{3!} (x - 1)^3 = \frac{c}{6} (x - 1)^3$, for some $c = c_x$ lying strictly between 1 and $x$.

1. C) Write $f(x) = T_2(x) + R_2(x)$, then

$$\frac{f(x) - e(x + 1)}{x - 1} = \frac{e + e(x - 1) + \frac{e}{2} (x - 1)^2 + \frac{c}{6} (x - 1)^3 - e - cx}{x - 1}$$

$$= \frac{-e}{x - 1} + \frac{e}{2} (x - 1) + \frac{c}{6} (x - 1)^2.$$

The latter two terms approach 0 as $x \to 1$. However, $\lim_{x \to 1^-} \frac{-e}{x - 1} = -\infty$ and $\lim_{x \to 1^+} \frac{-e}{x - 1} = \infty$. Hence the limit diverges.

2. A) Write $\epsilon_n = x_n - a$. For bisection, $|\epsilon_{n+1}| \leq \frac{1}{2} |\epsilon_n|$, the order is 1 and the rate is 1/2. For the interval $[\frac{1}{2}, \frac{3}{2}]$ only 1 step is needed.

2. B)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 1}{2x_n}$$

$$= x_n - \frac{x_n}{2} + \frac{1}{2x_n}$$

$$= \frac{1}{2}(x_n + 1/x_n).$$

The cobweb diagram suggests that iterations converge to 1 for any $x_0 > 1$. In particular, $x_0 = 3/2$ yields convergence. Since $f''(1) \neq 0$, the order of convergence is 2. The rate is $\left| \frac{f''(1)}{2f'(1)} \right| = 1/2$. 
Alternative argument: write $x_n = 1 + \epsilon_n$ and assume that the errors $\epsilon_n$ tend to 0 as $n \to \infty$. Then

$$1 + \epsilon_{n+1} = \frac{1}{2} \left( 1 + \epsilon_n + \frac{1}{1 + \epsilon_n} \right)$$

$$= \frac{1}{2} \left( 1 + \epsilon_n + 1 - \epsilon_n + \epsilon_n^2 - \ldots \right)$$

$$= 1 + \frac{1}{2} \epsilon_n^2 - \ldots .$$

It follows that $\epsilon_{n+1} \approx \frac{1}{2} \epsilon_n^2$ (terms of order 3 and higher neglected).

2. C)

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

$$= x_n - (x_n^2 - 1) \frac{x_n - x_{n-1}}{x_n^2 - x_{n-1}^2}$$

$$= x_n - \frac{x_n^2 - 1}{x_n + x_{n-1}}$$

$$= \frac{1}{x_n + x_{n-1}} .$$

The cobweb diagram suggests that $x_0 = 1/2$, $x_1 = 3/2$ yield convergence.

Since $f'(1) \neq 0$, the order is $\frac{1}{2}(1 + \sqrt{5})$. The rate is $\left| \frac{f''(1)}{2f'(1)} \right| \sqrt{5} = 2^{1/2} \sqrt{5}$.  

3. A) Passing to the limit in the equation $x_{n+1} = x_n - \arctan(x_n)$:

$$\alpha = \alpha - \arctan(\alpha)$$

$$\arctan(\alpha) = 0$$

$$\alpha = 0 .$$

3. B) Check the assumptions of Theorem 3.4.2. First, $g(x) = x - \arctan x$ is continuously differentiable for all $x$: $g'(x) = 1 - \frac{1}{1 + x^2}$ is continuous.

Second, $g(x)$ takes any interval $[-K, K]$ into $[-K, K]$. Reason: $g(x)$ is an odd function and $0 < g(x) < x$ for any $x > 0$, because $0 < \arctan x < x$ for $x > 0$. Third,

$$\max_{|x| \leq K} |g'(x)| = 1 - \frac{1}{K^2 + 1} < 1 .$$

Therefore any $K > 0$ will do.

3. C)

$$(\arctan x)' = \frac{1}{1 + x^2} = 1 - x^2 + \ldots , \quad |x| < 1,$$

$$\arctan x = x - \frac{1}{3} x^3 + \ldots , \quad |x| < 1,$$

$$g(x) = x - \arctan x = \frac{1}{3} x^3 + \ldots , \quad |x| < 1 .$$

Hence $x_{n+1} \approx \frac{1}{3} x_n^3$, which shows that the order is 3 and the rate is 1/3.