LINEAR EQUATIONS OF HIGHER ORDER. EXAMPLES.

GENERAL FRAMEWORK

We consider linear ODE of order $n$:

\[
x^{(n)}(t) + P_{n-1}(t)x^{(n-1)}(t) + \cdots + P_1(t)x'(t) + P_0(t)x(t) = 0
\]

and

\[
x^{(n)}(t) + P_{n-1}(t)x^{(n-1)}(t) + \cdots + P_1(t)x'(t) + P_0(t)x(t) = f(t)
\]

on the interval $\alpha < t < \beta$. All functions $P_k(t), f(t)$ are assumed to be continuous on $(\alpha, \beta)$.

If $\phi_1(t), \ldots, \phi_m(t)$ are solutions of (1) then any linear combination $c_1\phi_1 + \cdots + c_m\phi_m$ is also a solution of (1) (superposition principle). If $m = n$ and $\phi_1(t), \ldots, \phi_n(t)$ are linearly independent then

\[x(t) = c_1\phi_1 + \cdots + c_n\phi_n, \quad \alpha < t < \beta,
\]

is the general solution of (1).

The general solution of (2) has the form

\[x(t) = x_p(t) + x_c(t), \quad \alpha < t < \beta,
\]

where $x_p(t)$ is some solution of (2) (a particular solution) and $x_c(t)$ is the general solution of (1) (complementary function).

If $x_1(t)$ is a solution of (2) for $f = f_1(t)$ and if $x_2(t)$ is a solution of (2) for $f = f_2(t)$, then $x_1(t) + x_2(t)$ is a solution of (2) with right hand side $f = f_1(t) + f_2(t)$. This superposition principle is helpful when we are looking for a particular solution of (2) where $f(t)$ is a sum of different functions.

CONSTANT COEFFICIENTS.

If all coefficients $P_k(t)$ are constant, (1) takes the form

\[
x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \cdots + a_1x'(t) + a_0x(t) = 0,
\]

where $a_0, \ldots, a_{n-1}$ are real numbers. Solutions of (3) are linear combinations of exponential functions (with real or complex exponents).

The assertion that $x = e^{rt}$ satisfies (3) leads to the indicial equation for $r$:

\[r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0.
\]
There is a total of \( n \) roots of (4) listed with multiplicities (roots could be tricky to find). Because \( a_k \) are real, any complex roots will come in conjugate pairs \( \lambda \pm i\mu \).

The general solution of (3) has the following structure. Every real root \( r \) of multiplicity \( p \) (\( 1 \leq p \leq n \)) contributes \( p \) functions
\[
e^{rt}, \; te^{rt}, \ldots, t^{p-1}e^{rt}.
\]
In particular, a simple root \( (p = 1) \) brings in a single exponential \( e^{rt} \).

Every pair of complex conjugate roots \( r = \lambda \pm i\mu \) of multiplicity \( p \) (\( 2 \leq p \leq n \)) contributes \( 2p \) functions
\[
e^{\lambda t} \cos(\mu t), \; te^{\lambda t} \cos(\mu t), \ldots, t^{p-1}e^{\lambda t} \cos(\mu t),
\]
and
\[
e^{\lambda t} \sin(\mu t), \; te^{\lambda t} \sin(\mu t), \ldots, t^{p-1}e^{\lambda t} \sin(\mu t).
\]
In particular, a simple pair \( (p = 1) \) brings in two solutions \( e^{\lambda t} \cos(\mu t) \) and \( e^{\lambda t} \sin(\mu t) \). The resulting collection of \( n \) functions is linearly independent.

**Example.** Suppose that \( n = 13 \) and (4) has the roots
\[-1, \; 0, \; 0, \; 1, \; 1, \; \pm 2i, \; -3 \pm 2i, \; -3 \pm 2i, \; -3 \pm 2i.\]
Then the general solution of (3) is
\[
c_1e^{-t} + [c_2 + c_3t]e^t + [c_4 + c_5t]t^2 + [c_6 \cos(2t) + c_7 \sin(2t)] + e^{-3t}[(c_8 + c_9t + c_{10}t^2) \cos(2t) + (c_{11} + c_{12}t + c_{13}t^2) \sin(2t)].
\]

**Example.** Check that the general solution of \( x^{(4)} + 2x^{(2)} + x = 0 \) is given by
\[x = (a_1 + a_2t) \cos t + (b_1 + b_2t) \sin t.
\]

**Linear Independence and the Wronskian.**

The functions \( \phi_1, \ldots, \phi_n \) are linearly independent on \((\alpha, \beta)\) if for any constants \( c_1, \ldots, c_n \), not all 0, the linear combination \( c_1\phi_1 + \cdots + c_n\phi_n \) is not identically zero on \((\alpha, \beta)\). If for some choice of \( c_1, \ldots, c_n \), not all 0, we have
\[c_1\phi_1 + \cdots + c_n\phi_n \equiv 0, \quad \alpha < t < \beta,
\]
the functions are linearly dependent.

**Example.** The functions \( 1, t, t^2 \) are linearly independent on any interval. Can you give a proof?

**Example.** The functions \( 1, \sin^2 x, \cos^2 x \) are linearly dependent on any interval:
\[1 \cdot \sin^2 t + 1 \cdot \cos^2 t + (-1) \cdot 1 = 0.
\]
Notice, however, that any two functions in this collection (say, 1 and \( \sin^2 x \)) are independent.
The Wronskian of $\phi_1, \ldots, \phi_n$ is the $n \times n$ determinant

$$W(t) = \begin{vmatrix} \phi_1 & \phi_2 & \ldots & \phi_n \\ \dot{\phi}_1 & \dot{\phi}_2 & \ldots & \dot{\phi}_n \\ \ddots & \ddots & \ddots & \ddots \\ \phi_{(n-1)} & \phi_{(n-1)} & \ldots & \phi_{(n-1)} \end{vmatrix}.$$ 

We can check functions for linear dependence on $(\alpha, \beta)$ by computing their Wronskian. This is because

$\phi_1, \ldots, \phi_n$ dependent $\Rightarrow W(t) \equiv 0$

$W(t_0) \neq 0$ $\Rightarrow$ $\phi_1, \ldots, \phi_n$ independent.

If $\phi_1, \ldots, \phi_n$ are solutions of (1), we have Abel's formula:

$$W(t) = W(t_0) \exp \left\{ -\int_{t_0}^{t} P_{n-1}(s) ds \right\}$$

for any $\alpha < t_0 < \beta$. Thus, for solutions of (1), the Wronskian either is identically zero or nonzero at all.

**HOW TO COMPUTE DETERMINANTS?**

Determinants may be computed by a reduction process, called the expansion of the determinant along the first row:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$

and so on. One can expand along any row or column (the more zero entries it contains the better) with the same outcome.

Here are several tricks for computing determinants. If two rows (columns) are equal then the determinant is 0. If one row (column) is a linear combination of some other rows (columns), the determinant is 0. If all entries below (above) the main diagonal are zero, the determinant is the product of just the diagonal entries. If you multiply a row (column) by a constant, the determinant scales by the same constant. If you interchange two rows (columns), the determinant changes sign. Adding a multiple of
one row (column) to another row (column) does not change the determinant.

Examples.

\[
\begin{pmatrix}
0 & 1 & 3 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{pmatrix}
= 0,
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 2
\end{pmatrix}
= 2,
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
= 0
\]

\[
\begin{pmatrix}
t & \cos t & \sin t \\
1 & -\sin t & \cos t \\
0 & -\cos t & -\sin t
\end{pmatrix}
= \begin{pmatrix} t \sin t & \cos t \\
-\sin t & \cos t + \sin t \end{pmatrix}
= t(\sin^2 t + \cos^2 t) = t,
\]

The method of undetermined coefficients.

The point of the method is to find a particular solution of (2). One looks for a function \(x_p(t)\) that has the same form as \(f(t)\). General guidelines are given on page 181.

Example. Let \(\ddot{x} - \dot{x} = t\). To find the general solution solve the homogeneous equation \(\ddot{x} - \dot{x} = 0\) first. The roots of the indicial equation \(r^3 - r = 0\) are \(r = 0, \pm 1\). Hence \(x_c(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{-t}\). Next, since the right side \(f(t) = t\) is a polynomial, we’ll look for a polynomial \(x_p(t)\). Note that \(x_p(t)\) cannot have the same degree 1 as \(t\) because there is no \(x\) term on the left and differentiation lowers the degree. But the choice \(x = at^2\) is successful: \(0 - 2at = t\) gives \(a = -1/2\). The general solution is therefore \(x = -\frac{1}{2}t^2 + c_1 e^{t} + c_2 e^{-t} + 3e^{-t}\).

Example. Let \(y''' - y'' - y' + y = 2x + 3e^x\). The indicial equation \(r^3 - r^2 - r + 1 = (r - 1)^2(r + 1) = 0\) has roots \(r = -1, 1, 1\). This gives \(y_c(x) = c_1 e^{-x} + c_2 e^x + c_3 xe^x\). Next, the right side \(f(x)\) is a sum of a polynomial and an exponential. By superposition a particular solution of \(y''' - y'' - y' + y = 2x + 3e^x\) can be found by summing a solution of \(y''' - y'' - y' + y = 2x\) and a solution of \(y''' - y'' - y' + y = 3e^x\). Plugging \(y = A_1 x + A_2\) into \(y''' - y'' - y' + y = 2x\) gives \(A_1 = A_2 = 2\). To find a solution of \(y''' - y'' - y' + y = 3e^x\) note that the exponential term \(e^x\) is a solution of the homogeneous equation and that so is \(xe^x\). So we’ll look for \(y\) in the form \(y = Bxe^x\). Plugging \(y = Bxe^x, y' = 2Bxe^x + Bx^2 e^x, y'' = 2Bxe^x + 4Bxe^x + Bx^2 e^x, y''' = 6Bxe^x + 6Bxe^x + Bx^2 e^x\) into \(y''' - y'' - y' + y = 3e^x\) we find that \(4Be^x = 3e^x\) or \(B = 3/4\). Therefore \(y_p = 2(x + 1) + \frac{3}{4} x^2 e^x\) is a particular solution of the equation \(y''' - y'' - y' + y = 2x + 3e^x\). The general solution is given by \(y = 2(x + 1) + \frac{3}{4} x^2 e^x + c_1 e^{-x} + c_2 e^x + c_3 xe^x\).
Variation of parameters.

This method is also aimed at finding a particular solution of (2). Let \( \phi_1(t), \ldots, \phi_n(t) \) be a known linearly independent set of solutions of (1). Look for \( x_p(t) \) in the form

\[
x_p = u_1 \phi_1 + \cdots + u_n \phi_n,
\]

where \( u_1, \ldots, u_n \) are functions to be determined.

To simplify differentiation (we want \( x_p \) to satisfy (2)), one requires that

\[
\begin{align*}
\dot{u}_1 \phi + \cdots + \dot{u}_n \phi_n &= 0 \\
\ddot{u}_1 \phi + \cdots + \ddot{u}_n \phi_n &= 0 \\
\dddot{u}_1 \phi + \cdots + \dddot{u}_n \phi_n &= 0 \\
&\quad \vdots \\
\dddot{u}_1 \phi^{(n-2)} + \cdots + \dddot{u}_n \phi^{(n-2)} &= 0.
\end{align*}
\]

It then follows from (2) that

\[
\dot{u}_1 \phi^{(n-1)} + \cdots + \dot{u}_n \phi_n^{(n-1)} = f(t).
\]

The \( n = (n - 1) + 1 \) equations together allow to find \( \dot{u}_1, \ldots, \dot{u}_n \). The functions \( u_1, \ldots, u_n \) are then recovered by integration.

There are determinant representations for \( \dot{u}_k \):

\[
\dot{u}_k(t) = f(t) \frac{W_k(t)}{W(t)}, \quad k = 1, \ldots, n,
\]

where \( W \) is the Wronskian of \( \phi_1(t), \ldots, \phi_n(t) \) and \( W_k \) is obtained from \( W \) by replacing the the \( k \)-th column by the column

\[
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix}.
\]

There is also nothing wrong in solving the system directly.

**Example.** Let’s find a solution of \( \ddot{x} - \dot{x} = t \) using variation of parameters.

We can take

\[
\phi_1 = 1, \quad \phi_2 = e^t, \quad \phi_3 = e^{-t}.
\]

Write \( x_p = u_1 + u_2 e^t + u_3 e^{-t} \). The following conditions must be met:

\[
\begin{align*}
\dot{u}_1 + \dot{u}_2 e^t + \dot{u}_3 e^{-t} &= 0 \\
0 + \dot{u}_2 e^t - \dot{u}_3 e^{-t} &= 0 \\
0 + \ddot{u}_2 e^t + \ddot{u}_3 e^{-t} &= t.
\end{align*}
\]
The system easily yields $2\dot{u}_2 e^t = t$, $2\dot{u}_3 e^{-t} = t$, and $\dot{u}_1 + t = 0$. Hence

$u_1 = -\frac{1}{2}t^2 + c$, $u_2 = \frac{1}{2} \int te^{-t}dt = -\frac{1}{2}(t + 1)e^{-t} + c,$

$u_3 = \frac{1}{2} \int te^t dt = \frac{1}{2}(t - 1)e^t + c$. Consequently, one choice for $x_p$ is

$x_p = -\frac{1}{2}t^2 - \frac{1}{2}(t + 1) + \frac{1}{2}(t - 1) = -\frac{1}{2}t^2 - 1.$