Vectors & Vector Arithmetic

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 11.2 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to perform arithmetic operations on vectors and understand the geometric consequences of the operations.
- Know how to compute the magnitude of a vector and normalize a vector.
- Be able to use vectors in the context of geometry and force problems.

PRACTICE PROBLEMS:

1. Find the components of vector $\vec{v}$ and sketch an equivalent vector with its initial point at the origin.

$$\vec{v} = \langle 5, 3 \rangle$$
2. Sketch the vector $\vec{u} + \vec{v} + \vec{w}$ and express it in component form.

3. The figure below is a parallelogram. Express $\vec{w}$ in terms of $\vec{u}$ and $\vec{v}$.

$$\vec{w} = \frac{1}{2} (\vec{u} + \vec{v})$$
4. Consider the points $P_1(2, 3)$ and $P_2 = (5, -1)$. Find the components of the vector $\overrightarrow{P_1P_2}$. Sketch $P_1$, $P_2$, $\overrightarrow{P_1P_2}$, and an equivalent vector with its initial point at the origin.

5. Consider the points $P_1(1, 2, 3)$ and $P_2(5, 4, 6)$. Find the components of the vector $\overrightarrow{P_1P_2}$.

6. Let $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{i} + 4\mathbf{j} + \mathbf{k}$. Compute each of the following:

   (a) $2\mathbf{u} - 3\mathbf{w}$
   \[-15\mathbf{i} - 8\mathbf{j} - 5\mathbf{k}\]

   (b) $\|\mathbf{u} + \mathbf{v}\|
   \sqrt{41}$

   (c) $\|\mathbf{u}\| + \|\mathbf{v}\|
   \sqrt{14} + \sqrt{29}$

   (d) $\|2\mathbf{u}\|
   2\sqrt{14}$

   (e) $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$
   $\frac{1}{\mathbf{v}}$

7. For each of the following, find a vector which satisfies the given conditions.

   (a) A unit vector which is in the opposite direction of $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$
   $\frac{-3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$; Detailed Solution: [Here]
(b) A unit vector which is in the same direction as the vector from \( P_1(1, 0, 5) \) to \( P_2(3, -1, 2) \)
\[
\left\langle \frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \right\rangle; \text{ Video Solution: Here}
\]

(c) A vector which is in the opposite direction of \( \vec{v} = (1, 2, 3) \) and whose magnitude is half that of \( \vec{v} \).
\[
\left\langle -\frac{1}{2}, -1, -\frac{3}{2} \right\rangle; \text{ Detailed Solution: Here}
\]

(d) A vector which is in the same direction of \( \vec{w} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \) and which has a length of \( \sqrt{5} \)
\[
\frac{\sqrt{5}}{\sqrt{14}} \mathbf{i} - \frac{2\sqrt{5}}{\sqrt{14}} \mathbf{j} + \frac{3\sqrt{5}}{\sqrt{14}} \mathbf{k}; \text{ Detailed Solution: Here}
\]

(e) A vector in 2-space which makes an angle of \( \theta = \frac{\pi}{6} \) with the positive \( x \)-axis and which has a magnitude of 4.
\[
\left\langle 2\sqrt{3}, 2 \right\rangle; \text{ Detailed Solution: Here}
\]

(f) A vector in 2-space which makes an angle of \( \theta = 210^\circ \) with the positive \( x \)-axis and which has a length of 2.
\[
\left\langle -\sqrt{3}, -1 \right\rangle; \text{ Detailed Solution: Here}
\]

8. Find the value(s) of \( a \) so that the vectors \( \vec{v} = \langle a^2, 6 \rangle \) and \( \vec{w} = \langle 4a, 2 \rangle \) are parallel.
\[ a = 0 \text{ or } a = 12 \]

9. Vectors \( \vec{v} \) and \( \vec{w} \), shown below, are unit vectors. Find the components of \( \vec{v} + \vec{w} \).

\[
\left\langle \frac{\sqrt{2} + \sqrt{3}}{2}, \frac{\sqrt{2} - 1}{2} \right\rangle; \text{ Detailed Solution: Here}
\]
10. For each of the following, find the magnitude of the resultant force and the angle that it makes with the positive x-axis.

(a) The magnitude is \(20\sqrt{5}\) lb at an angle of \(\pi - \tan^{-1}\left(\frac{1}{2}\right)\) radians counterclockwise with the positive x-axis.

(b) The magnitude is \(150\sqrt{10}\) N at an angle of \(\tan^{-1}\left(\frac{1}{2}\right)\) radians counterclockwise with the positive x-axis.
11. A weight of 200 Newtons (N) is being supported by two wires, as shown below. Find the tension in each wire.

Let $F_1$ be the wire which makes an angle of $45^\circ$ clockwise with the ceiling and $F_2$ be the wire which makes an angle of $30^\circ$ counterclockwise with the ceiling. Then

\[ \|F_2\| = \frac{400}{1 + \sqrt{3}} \text{ N and } \|F_1\| = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{400}{1 + \sqrt{3}} \text{ N.} \]