Chapter 1.5 Practice Problems

EXPECTED SKILLS:

- Know what it means for a function to be continuous at a specific value and on an interval.
- Find values where a function is not continuous; specifically, you should be able to do this for polynomials, rational functions, exponential and logarithmic functions, and other elementary functions.
- Determine the values for which a piecewise function is discontinuous, if any such values exist.
- Use the Intermediate Value Theorem to show the existence of a solution to an equation.

PRACTICE PROBLEMS:

Use the graph of $f(x)$, shown below, to answer questions 1-3

1. For which values of $x$ is $f(x)$ discontinuous?

   $f(x)$ is discontinuous when $x = 0$, $x = 3$, and $x = 6$.

2. At each point of discontinuity, explain why $f(x)$ is discontinuous.

   - At $x = 0$, $f(x)$ is discontinuous because $\lim_{x \to 0} f(x)$ DNE.
   - At $x = 3$, $f(x)$ is discontinuous because $\lim_{x \to 3} f(x) \neq f(3)$.
   - At $x = 6$, $f(x)$ is discontinuous because $f(6)$ is undefined.
3. Determine whether $f(x)$ is continuous on the given interval. If not, explain why.

(a) $[-8, -4]$
Yes

(b) $[-8, 0]$
No because $\lim_{x \to 0^-} f(x) \neq f(0)$

(c) $[-8, 0)$
Yes

(d) $[-2, 1]$
No because $\lim_{x \to 0} f(x)$ DNE

(e) $(3, 6)$
Yes

(f) $[3, 6)$
No because $\lim_{x \to 3^+} f(x) \neq f(3)$

(g) $(6, 9]$
Yes

(h) $(6, 9]$
No because $f(6)$ is undefined

4. For each of the following, sketch the graph of a function, $y = f(x)$, which satisfies the given characteristic. (There are many possible answers for each)

(a) $f(x)$ is continuous everywhere except at $x = 1$.
Any graph for which either $f(1)$ is undefined or $\lim_{x \to 1^-} f(x)$ DNE or $\lim_{x \to 1^+} f(x) \neq f(1)$

(b) $f(x)$ is continuous everywhere except at $x = -2$ where the $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x)$.
Any graph for which either $f(-2)$ is undefined or $\lim_{x \to -2} f(x) \neq f(-2)$

(c) $f(x)$ is continuous everywhere except at $x = 0$, where $f(0) = 2$.
Any graph for which $\lim_{x \to 0^-} f(x)$ DNE or $\lim_{x \to 0^+} f(x) \neq 2$
5. Sketch the graph of a function which satisfies the following criteria:

- The domain of \( f(x) \) is \([1, 3]\)
- \( f(x) \) is continuous on \([1, 2]\) and \((2, 3]\).
- \( f(x) \) is not continuous on \([1, 3]\)

![Graph of f(x)](image)

Determine the values of \( x \) where the given function is discontinuous, if any such values exist.

6. \( f(x) = |x| \)

   \( f(x) \) is always continuous

7. \( f(x) = x^2 - x - 5 \)

   \( f(x) \) is always continuous

8. \( f(x) = \frac{x}{x-1} \)

   \( f(x) \) is discontinuous when \( x = 1 \)

9. \( f(x) = \sqrt{x - 1} \)

   \( f(x) \) is always continuous

10. \( f(x) = \frac{x^2-1}{x-1} \)

    \( f(x) \) is discontinuous when \( x = 1 \)
11. \( f(x) = \frac{5}{2x} + \frac{x^2 + 9x + 18}{x+3} \)

\( f(x) \) is discontinuous when \( x = 0 \) and when \( x = -3 \)

12. \( f(x) = \frac{x^2 - 4}{x-2} \)

\( f(x) \) is discontinuous when \( x = 2 \)

13. \( f(x) = \frac{x^2 + 3x - 10}{x-7} \)

\( f(x) \) is discontinuous when \( x = 7 \)

14. \( f(x) = \frac{1}{x^2 - 2} + \frac{x^3 - 1}{2x^2 - 1} \)

\( f(x) \) is discontinuous when \( x = \sqrt{2}, x = -\sqrt{2}, x = \frac{\sqrt{2}}{2}, \) and \( x = -\frac{\sqrt{2}}{2} \)

15. \( f(x) = \begin{cases} x^2 - 1, & \text{if } x < 2 \\ \frac{3}{x-1}, & \text{if } x \geq 2 \end{cases} \)

\( f(x) \) is always continuous

16. \( f(x) = \begin{cases} 5 + \frac{1}{x}, & \text{if } x < -1 \\ 3x^2 + 2x + 3, & \text{if } x \geq -1 \end{cases} \)

\( f(x) \) is always continuous

17. \( f(x) = \begin{cases} x^2 - 3x + 4, & \text{if } x \leq 1 \\ x^4 - 4x^3 - 2x^2 + 6, & \text{if } x > 1 \end{cases} \)

\( f(x) \) is discontinuous when \( x = 1 \)

18. Find the value(s) of \( k \) such that \( f(x) \) is continuous everywhere:

\[ f(x) = \begin{cases} x^2 - 7, & \text{if } x \leq 2 \\ 4x^3 - 3kx + 2, & \text{if } x > 2 \end{cases} \]

\[ k = \frac{37}{6} \]
19. Find the value(s) of $k$ and $m$ such that $f(x)$ is continuous everywhere:

$$f(x) = \begin{cases} 
2x + 8m, & \text{if } x \leq -2 \\
mx + k, & \text{if } -2 < x \leq 2 \\
-3x^2 + 8x - 2k, & \text{if } x > 2 
\end{cases}$$

$m = \frac{1}{2}$ and $k = 1$

Use the following definitions to answer question 20

- A function $f(x)$ has a **removable discontinuity** at $x = a$ if $\lim_{x \to a} f(x)$ exists but $f(x)$ is not continuous at $x = a$. This could be because $f(a)$ is undefined or because $\lim_{x \to a} f(x) \neq f(a)$.

- A function $f(x)$ has a **jump discontinuity** at $x = a$ if $\lim_{x \to a^-} f(x)$ exists and $\lim_{x \to a^+} f(x)$ exists, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

20. For problems 6-17, if $f(x)$ has a discontinuity at $x = a$, determine whether it is a removable discontinuity, a jump discontinuity, or neither.

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<td>Problem 8</td>
<td>The discontinuity is neither</td>
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<td>Problem 9</td>
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<td>Problem 10</td>
<td>$x = 1$ is a removable discontinuity. Note: $\lim_{x \to 1} \left( \frac{x^2 - 1}{x - 1} \right) = 2$</td>
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<td>Problem 11</td>
<td>$x = 0$ is neither; $x = -3$ is a removable discontinuity. Note: $\lim_{x \to -3} \left( \frac{x^2 + 9x + 18}{x + 3} \right) = 3$</td>
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<td>Problem 12</td>
<td>$x = 2$ is a removable discontinuity; Note $\lim_{x \to 2} \left( \frac{x^2 - 4}{x - 2} \right) = 4$</td>
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<td>Problem 13</td>
<td>$x = 7$ is neither</td>
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<td>Problem 17</td>
<td>$x = 1$ is a jump discontinuity</td>
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21. Show that the following equation $x^3 - x^2 + 3x - 1 = 1$ has at least one solution in $(0, 1)$.

Let $f(x) = x^3 - x^2 + 3x - 2$. It suffices to show that there exists a $c$ in $(0, 1)$ such that $f(c) = 0$. Since $f(x)$ is a polynomial, it is continuous everywhere on $(-\infty, \infty)$. Specifically, it is continuous on $[0, 1]$. Since $f(0) = -2 < 0$ and $f(1) = 1 > 0$, the Intermediate Value Theorem states that there exists some $c \in (0, 1)$, $f(c) = 0$. The result follows.
22. Show that $f(x) = x^3 - 9x + 5$ has at least one $x$-intercept in $(1,10)$.

We need to show that there exists at least one solution to $f(x) = 0$. Since $f(x)$ is a polynomial, it is continuous on $[1,10]$. Notice that $f(1) = -3 < 0$ and $f(10) = 915 > 0$. Thus, the Intermediate Value Theorem states that there must be a $c$ in $(1,10)$ with $f(c) = 0$.

23. Use the intermediate value theorem to show that $x^3 - 2x^2 - 2x + 1 = 0$ has at least TWO solutions in $[0,5]$.

We will apply the IVT twice first on $[0,1]$ and then on $[1,5]$. Let $f(x) = x^3 - 2x^2 - 2x + 1$. Since $f(x)$ is a polynomial, it is continuous on $(-\infty, \infty)$. As a result, it is continuous on $[0,1]$ and $[1,5]$. Notice that $f(0) = 1 > 0$ and $f(1) = -2 < 0$. So, the IVT implies that there exists a $c$ in $(0,1)$ such that $f(c) = 0$. Similarly, notice that $f(1) = -2 < 0$ and $f(5) = 66 > 0$. So, the IVT implies that there exists a $d$ in $(1,5)$ such that $f(d) = 0$. 