1. (10 points) Use a double integral to find the volume of the solid that is bounded above by the plane \( z = 4 - x - y \) and below by the rectangle 
\[ R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 2\} \]. You may choose whichever order of integration you prefer.

**Method One:**
\[
\int_0^1 \int_0^2 (4-x-y) \, dy \, dx = \int_0^1 \left[ (4-x)y \right]_0^2 - \frac{1}{2} y^2 \bigg|_0^2 \, dx \\
= \int_0^1 (4-2x) - \frac{1}{2} \cdot 2^2 \, dx \\
= 6x \bigg|_0^1 - 2 \cdot \frac{1}{2} x^2 \bigg|_0^1 = 6 - 1 = \sqrt{5}
\]

**Method Two:**
\[
\int_0^2 \int_0^1 (4-x-y) \, dx \, dy = \int_0^2 \left[ (4-y)x \right]_0^1 - \frac{1}{2} x^2 \bigg|_0^1 \, dy \\
= \int_0^2 (4-y) - \frac{1}{2} \ , dy = \int_0^2 \left( \frac{7}{2} - y \right) \, dy \\
= \frac{7}{2} y \bigg|_0^2 - \frac{1}{2} y^2 \bigg|_0^2 = 7 - 2 = \sqrt{5}
\]
2. Consider the double integral \( \int \int_{R} xy^{2} \, dA \) where \( R \) is the region bounded by 
\[ y = 1, \quad y = 2, \quad x = 0, \quad \text{and} \quad y = x. \]

a. (10 points) Set this up as an iterated integral (or integrals), where \( dA = dy \, dx \).

\( \text{DO NOT EVALUATE THE INTEGRAL(S).} \)

b. (10 points) Set this up as an iterated integral (or integrals), where \( dA = dx \, dy \).

\( \text{DO NOT EVALUATE THE INTEGRAL(S).} \)

\[ \begin{align*}
\text{(a)} \quad \int_{0}^{1} \int_{0}^{x} xy^{2} \, dy \, dx & \quad + \quad \int_{1}^{2} \int_{x}^{2} xy^{2} \, dy \, dx \\
\text{(b)} \quad \int_{1}^{2} \int_{0}^{y} x \, y^{2} \, dx \, dy
\end{align*} \]
3. (15 points) Evaluate the integral by first reversing the order of integration.

\[ \int_0^2 \int_0^{x^2/2} e^{x^2} \, dy \, dx \]

First, find the new limits of integration.

The region of integration is bounded by the lines:
- \( y = 0 \)
- \( y = x^2/2 \)
- \( x = 1 \)
- \( x = 0 \)

The region can be described by the inequality:
\( 0 \leq y \leq x^2/2 \)
\( 0 \leq x \leq 1 \)

To reverse the order of integration, we integrate with respect to \( x \) first and then \( y \):

\[ \int_0^{x^2/2} \int_0^1 e^{x^2} \, dx \, dy \]

Evaluate the inner integral:
\[ \int_0^1 e^{x^2} \, dx \]

Make the substitution: \( u = x^2 \), \( du = 2x \, dx \), \( x = 0 \Rightarrow u = 0 \), \( x = 1 \Rightarrow u = 1 \)

\[ = \int_0^1 e^{u} \, \frac{du}{2} \]

\[ = 2 \left[ e^u \right]_0^1 \]
\[ = 2(e - 1) \]

So the integral evaluates to:
\[ 2(e - 1) \]
4. a. (10 points) Find an equation of the tangent plane to the surface 
\[ xz + 2yz^2 - z^3 = 7 \] at the point \((2,3,1)\).

b. (10 points) Find parametric equations for the tangent line to the curve of 
intersection of \(xz + 2yz^2 - z^3 = 7\) and \(-x + y - z = 0\) at \((2,3,1)\).

\[ f(x, y, z) = xz + 2yz^2 - z^3 \]

\[ f_x = z \quad f_x(2,3,1) = 1 \]
\[ f_y = 2z^2 \quad f_y(2,3,1) = 2 \]
\[ f_z = x + 4yz - 3z^2 \quad f_z(2,3,1) = 2 + 12 - 3 = 11 \]
\[ \nabla f(2,3,1) = \langle 1, 2, 11 \rangle \]

Tangent Plane: \(1(x-2) + 2(y-3) + 11(z-1) = 0\) \(\Rightarrow\) \(x + 2y + 11z = 19\)

\[ g(x, y, z) = -x + y - z \]

\[ g_x = -1 \quad g_y = 1 \quad g_z = -1 \]
\[ \nabla g(2,3,1) = \langle -1, 1, -1 \rangle \]
\[ \nabla f(2,3,1) \times \nabla g(2,3,1) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 11 \\ -1 & 1 & -1 \end{vmatrix} \]
\[ = \langle -2 - 11 \rangle \hat{k} - \langle -1 + 11 \rangle \hat{j} + \langle 1 + 2 \rangle \hat{k} \]
\[ = \langle -13, -10, 3 \rangle \]

Tangent Line: \[
\begin{cases}
  x = 2 - 13t \\
  y = 3 - 10t \\
  z = 1 + 3t
\end{cases}
\]
5. (15 points) Identify all the critical points of the given function. Then classify each critical point as a relative maximum, relative minimum, or saddle point.

\[ f(x, y) = xy - x^3 - y^2 \]

\[
\begin{align*}
\frac{\partial f}{\partial x} &= y - 3x^2 = 0 \implies y = 3x^2 \\
\frac{\partial f}{\partial y} &= x - 2y = 0 \implies x - 6x^2 = 0 \\
&= x(1 - 6x) = 0 \\
x &= 0 \quad x = \frac{1}{6} \\
y &= 0 \quad y = \frac{1}{12}
\end{align*}
\]

Critical Points: \((0, 0), \left(\frac{1}{6}, \frac{1}{12}\right)\)

\[
\begin{align*}
f_{xx} &= -6x \\
f_{yy} &= -2 \\
f_{xy} &= 1
\end{align*}
\]

\[
\Delta(x, y) = (-6x)(-2) - (1)^2 = 12x - 1
\]

\[
\begin{align*}
\Delta(0, 0) &= -1 < 0 \implies \text{saddle point at } (0, 0) \\
\Delta\left(\frac{1}{6}, \frac{1}{12}\right) &= 2 - 1 = 1 > 0 \\
f_{xx}\left(\frac{1}{6}, \frac{1}{12}\right) &= -1 < 0 \implies \text{relative maximum at } \left(\frac{1}{6}, \frac{1}{12}\right)
\end{align*}
\]
6. (20 points) Find the absolute maximum and minimum values of 
\[ f(x, y) = xy - x - y \] on the closed triangular region \( R \) with vertices 
(0, 0), (4, 0) and (0, 4).

\[ f_x = y - 1 = 0 \Rightarrow y = 1 \]
\[ f_y = x - 1 = 0 \Rightarrow x = 1 \]

**Critical Point:** (1, 1) \[ f(1, 1) = -1 \]

**Boundary:**

(1) From (0, 0) to (4, 0):
\[ y = 0, \ 0 \leq x \leq 4 \]
\[ f(x, 0) = -x = u(t) \]
\[ u'(t) = -1 \neq 0 \ \text{No critical points on} \ 0 < x < 4 \]

(2) From (0, 0) to (0, 4):
\[ x = 0, \ 0 \leq y \leq 4 \]
\[ f(0, y) = -y = v(s) \]
\[ v'(s) = -1 \neq 0 \ \text{No critical points on} \ 0 < y < 4 \]

(3) From (0, 4) to (4, 0):
\[ y = -x + 4, \ 0 \leq x \leq 4 \]
\[ f(x, -x+4) = x(-x+4) - x - (-x+4) = -x^2 + 4x - x + x - 4 = -x^2 + 4x - 4 = w(u) \]
\[ w'(u) = -2x + 4 = 0 \Rightarrow x = 2 \ \text{w(2) = 0} \]

**Vertices:**
\[ f(0, 0) = 0 \]
\[ f(4, 0) = -4 \]
\[ f(0, 4) = -4 \]

**Compare all values:**

Absolute Max: \( 0 \)
Absolute Min: \( -4 \)
7. (Bonus 5 points) **DON'T, DON'T, DON'T** spend time on the bonus problem unless you feel **VERY, VERY, VERY** confident with your answers on the rest of the test.

Consider the two surfaces described in problem 4:

\[
\begin{align*}
1 & \quad xz + 2yz^2 - z^3 = 7 \\
2 & \quad -x + y - z = 0
\end{align*}
\]

(a) (3 points) The curve of intersection of these two surfaces can be parameterized in terms of the variable \( z \). Find this parameterization in terms of \( z \).

(b) (2 points) Find the tangent vector to the curve of part (a), evaluated at the value \( z = 1 \). (The resulting vector should be a scalar multiple of the direction vector you found in problem (4b).

\[
\begin{align*}
\text{(a)} & \quad \text{From } 2: \quad y = x + z \quad \text{Plug into 1:} \quad xz + 2(x+z)z^2 - z^3 = 7 \\
& \quad \text{Solve for } x: \quad xz + 2xz^2 + 2z^3 - z^3 = 7 \quad x = \frac{7 - z^3}{z + 2z^2} \\
\text{From } 2: \quad x = y - z \quad \text{Plug into 1:} \quad (y-z)z + 2yz^2 - z^3 = 7 \\
& \quad \text{Solve for } y: \quad yz - z^2 + 2yz^2 - z^3 = 7 \quad y = \frac{7 + z^2 + z^3}{z + 2z^2}
\end{align*}
\]

Parameterization: \( x = \frac{7 - z^3}{z + 2z^2}, \quad y = \frac{7 + z^2 + z^3}{z + 2z^2}, \quad z = z \)

\[
\begin{align*}
\text{(b)} & \quad \frac{dx}{dz} = \frac{(z+2z^2)(-3z^2) - (7-z^3)(1+4z)}{(z+2z^2)^2} \quad \frac{dx}{dz} \bigg|_{z=1} &= -\frac{9-30}{9} = \frac{-39}{9} \\
\frac{dy}{dz} &= \frac{(z+2z^2)(2z+3z^2) - (7+z^2+z^3)(1+4z)}{(z+2z^2)^2} \quad \frac{dy}{dz} \bigg|_{z=1} &= \frac{15-45}{9} = \frac{-30}{9} \\
\frac{dz}{dz} &= 1
\end{align*}
\]

Tangent vector: \( \langle -\frac{39}{9}, -\frac{30}{9}, 1 \rangle \)

Note: \( \langle -\frac{39}{9}, -\frac{30}{9}, 1 \rangle = \frac{1}{3} \langle -13, -10, 3 \rangle \) from (4b)
This Page Blank for Extra Work