Show all your work on the exam paper, legibly and in detail, to receive full credit. The use of a calculator or any other electronic device is prohibited. You may only use techniques discussed to date in class. You must simplify all answers unless you are explicitly instructed not to.

(12 points) Find the equation to a sphere with center \((-2, 1, 3)\) that is tangent to the \(yz\)-plane.

\[
\text{Tangent to the } yz\text{-plane } \Rightarrow \text{radius } = 2
\]

\[
(x + 2)^2 + (y + 1)^2 + (z + 3)^2 = 4
\]

(13 points) Find a vector with length \(\sqrt{17}\) that points in the same direction as the vector from point \(A: (1, 2, 9)\) to point \(B: (8, 2, 3)\)

\[
\overrightarrow{AB} = \langle 8-1, 2-2, 3-9 \rangle = \langle 7, 0, -6 \rangle
\]

\[
|\overrightarrow{AB}| = \sqrt{7^2 + 0^2 + (-6)^2} = \sqrt{49 + 36} = \sqrt{85}
\]

Solution: \(\frac{\sqrt{17}}{\sqrt{85}} \langle 7, 0, -6 \rangle = \frac{1}{\sqrt{5}} \langle 7, 0, -6 \rangle\)
(12 points) Given that \( \mathbf{v} = i - 3j + 2k \) and \( \mathbf{w} = i + j \), find \( \|2\mathbf{v} - 3\mathbf{w}\| \) and \( \frac{1}{\|\mathbf{v}\|} \).

\[
2\mathbf{v} - 3\mathbf{w} = \langle 2, -6, 4 \rangle - \langle 3, 3, 0 \rangle = \langle -1, -9, 4 \rangle
\]

\[
\|2\mathbf{v} - 3\mathbf{w}\| = \sqrt{(2)^2 + (-9)^2 + 4^2} = \sqrt{1 + 81 + 16} = \sqrt{98} = 7\sqrt{2}
\]

\[
\frac{\mathbf{v}}{\|\mathbf{v}\|}
\]
is a unit vector so

\[
\left\| \frac{1}{\|\mathbf{v}\|} \mathbf{v} \right\| = 1
\]

(13 points) Given that \( \mathbf{v} = \langle -2, 1, 6 \rangle \) and \( \mathbf{b} = \langle 0, -2, 1 \rangle \), express \( \mathbf{v} \) as the sum of a vector parallel to \( \mathbf{b} \) and a vector orthogonal to \( \mathbf{b} \). Clearly indicate which one is parallel to \( \mathbf{b} \) and which one is orthogonal to \( \mathbf{b} \).

\[
\text{proj}_\mathbf{b} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \mathbf{b} = \frac{(-2)(0) + (1)(-2) + (6)(1)}{0^2 + (-2)^2 + 1^2} \langle 0, -2, 1 \rangle
\]

\[
= \frac{4}{5} \langle 0, -2, 1 \rangle = \langle 0, -\frac{8}{5}, \frac{4}{5} \rangle
\]

\[
\mathbf{v} - \text{proj}_\mathbf{b} \mathbf{v} = \langle -2, 1, 6 \rangle - \langle 0, -\frac{8}{5}, \frac{4}{5} \rangle = \langle -2, \frac{13}{5}, \frac{26}{5} \rangle
\]

So

\[
\mathbf{v} = \langle 0, -\frac{8}{5}, \frac{4}{5} \rangle + \langle -2, \frac{13}{5}, \frac{26}{5} \rangle
\]

parallel to \( \mathbf{b} \) \hspace{1cm} \text{orthogonal to } \mathbf{b}
Consider the following lines:

\[ L_1 : x = 1 + t, \ y = 1 - t, \ z = 2t \]
\[ L_2 : x = 2 - t, \ y = t, \ z = 4 \]

(a) (10 points) Show that the lines intersect and find their point of intersection.
(b) (10 points) Find the equation of the plane that contains both lines.

(a)  
1. \[ 1 + t_1 = 2 - t_2 \]  
   By (3): \( t_1 = -2 \)
2. \[ 1 - t_1 = t_2 \]  
   By (2): \( 1 - 2 = t_2 \)
3. \[ 2t_1 = 4 \]  
   \( \Rightarrow t_2 = -1 \)

Plug in to (1): \( 1 + 2 = 2 - (2-1) \)
\[ 3 = 3 \]  

So lines intersect at \( (3, -1, 4) \)

(b) \( L_1 \parallel \vec{v}_1 = \langle 1, -1, 2 \rangle, \ L_2 \parallel \vec{v}_2 = \langle -1, 1, 0 \rangle \)

A normal vector to the requested plane is

\[ \vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = \langle 0 - 2, - (0 + 2), 1 - 1 \rangle = \langle -2, -2, 0 \rangle \]

Plane:
\[-2(x - 3) - 2(y + 1) + 0(z - 4) = 0\]
\[-2x + 6 - 2y - 2 = 0\]
\[-2x - 2y = -4\]
\[x + y = 2\]
Consider the following planes:

\[ x + y + z = 1 \]
\[ x + 2y + 2z = 1 \]

(a) (6 points) Show that the two planes are not parallel.
(b) (7 points) Find the acute angle of intersection between the two planes. You may leave the angle in the form of an inverse trigonometric function.
(c) (7 points) Find the line of intersection of the two planes.

(a) \( \vec{n}_1 = \langle 1, 1, 1 \rangle \perp x + y + z = 1 \)
\[ \vec{n}_2 = \langle 1, 2, 2 \rangle \perp x + 2y + 2z = 1 \]
\( \vec{n}_1, \vec{n}_2 \) are not scalar multiples of each other
So \( \vec{n}_1 \parallel \vec{n}_2 \), so the planes are not parallel

(b) \[ \cos \Theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\| \vec{n}_1 \| \| \vec{n}_2 \|} = \frac{(1)(1) + (1)(2) + (1)(2)}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{1^2 + 2^2 + 2^2}} = \frac{5}{3\sqrt{3}} \]
\[ \Theta = \arccos \left( \frac{5}{3\sqrt{3}} \right) \]

(c) See page 6
(10 points) Where does the line \( x = 1 + t, y = 3 - t, z = 2t \) intersect the cylinder \( x^2 + y^2 = 16 \)?

\[
(1+t)^2 + (3-t)^2 = 16
\]

\[
1 + 2t + t^2 + 9 - 6t + t^2 = 16
\]

\[
2t^2 - 4t - 6 = 0
\]

\[
t^2 - 2t - 3 = 0
\]

\[
(t - 3)(t + 1) = 0
\]

\[
t = 3 \quad t = -1
\]

\(t = 3: \ x = 1 + 3 = 4, \ y = 3 - 3 = 0, \ z = 2(3) = 6 \)

\(t = -1: \ x = 1 + (-1) = 0, \ y = 3 - (-1) = 4, \ z = 2(-1) = -2 \)

Two points of intersection: \((4, 0, 6)\) and \((0, 4, -2)\)
We need a point on the line of intersection, so we need a point on both planes. Let $z = 0$.

\[
\begin{align*}
x + y &= 1 \\
x + 2y &= 1
\end{align*}
\Rightarrow -y = 0 \Rightarrow y = 0 \Rightarrow x = 1
\]

So $(1, 0, 0)$ is on the line of intersection.

$\vec{n}_1 \times \vec{n}_2$ is parallel to the line of intersection.

\[
\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
1 & 1 & 1 \\
1 & 2 & 2
\end{vmatrix} = \vec{i} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = \langle 0, -1, 1 \rangle
\]

Line of intersection:

\[
\begin{align*}
x &= 1 + 0t = 1 \\
y &= 0 - 1t = -t \\
z &= 0 + 1t = t
\end{align*}
\]