Show all your work on the exam paper, legibly and in detail, to receive full credit. The use of a calculator or any other electronic device is prohibited. You may only use techniques discussed to date in class. You must simplify all answers unless you are explicitly instructed not to.

Consider the function \( z = x \ln \left(1 + y^2 \right) \).

a. (8 points) Suppose that \( x = x(t) \) and \( y = y(t) \) are differentiable functions. Use an appropriate form of the chain rule to find \( \frac{dz}{dt} \).

\[
(a) \quad \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

\[
\frac{dz}{dt} = \ln(1+y^2) \frac{dx}{dt} + \frac{2xy}{1+y^2} \frac{dy}{dt}
\]

b. (7 points) If \( x = 2 - t \) and \( y = \sqrt{t} \), use your answer in part (a) to find \( \frac{dz}{dt} \) at \( t = 1 \).

\[
\frac{dx}{dt} = -1 \quad \frac{dy}{dt} = \frac{1}{2\sqrt{t}}
\]

So \( \frac{dz}{dt} = \ln(1+t)(-1) + \frac{2(2-t)}{1+t} \left( \frac{1}{2\sqrt{t}} \right) \)

At \( t = 1 \): \( \frac{dz}{dt} = -\ln 2 + \frac{1}{2} \)
Consider the function \( f(x, y) = e^{x^2 y^2} \)

a. (8 points) Find the gradient of \( f \).

b. (7 points) Find the directional derivative of \( f \) in the direction of the vector \( v = \langle 1, 2 \rangle \) at the point \((0, -2)\).

c. (5 points) Find a unit vector \( u \) in the direction in which \( f \) decreases most rapidly at the point \((0, -2)\).

\[
(a) \quad f_x(x, y) = 2x e^{x^2 y^2}, \quad f_y(x, y) = 2 e^{x^2 y^2} y \\
\text{So } \nabla f = \left< 2x e^{x^2 y^2}, 2 e^{x^2 y^2} y \right>
\]

\[
(b) \quad \vec{v} = \left< 1, 2 \right> = \left< \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right> \text{ is a unit vector} \frac{\sqrt{1^2 + 2^2}}{\sqrt{5}} \\
in \text{ the direction of } \vec{v}. \text{ So} \\
D_{\vec{v}} f(0, -2) = \nabla f(0, -2) \cdot \vec{v} = \left< 0, -4 \right> \cdot \left< \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right> = -\frac{8}{\sqrt{5}}
\]

\[
(c) \quad -\nabla f(0, -2) = \left< 0, 4 \right> \text{ points in direction in which } f \text{ decreases most rapidly at } (0, -2) \\
\text{So } \vec{u} = \frac{\left< 0, 4 \right>}{\sqrt{0^2 + 4^2}} = \left< 0, 1 \right>
\]
Consider the surface \( \sin(xy) + \cos(xz) + \sin(yz) = 1 \).

a. (8 points) Find an equation for the tangent plane to the surface at the point \((\pi, 1, 0)\).

b. (4 points) Find the normal line to the surface at the point \((\pi, 1, 0)\). You may express your answer as a set of parametric equations or as a vector equation.

c. (8 points) Find the acute angle that the tangent plane to the surface at the point \((\pi, 1, 0)\) makes with the xy-plane. You may leave the angle in the form of an inverse trigonometric function.

\[
\begin{align*}
\mathbf{a} & = \mathbf{f}(x, y, z) = \sin(xy) + \cos(xz) + \sin(yz) \\
f_x(x, y, z) &= y \cos(xy) - z \sin(xz) \\
f_y(x, y, z) &= x \cos(xy) + z \cos(yz) \\
f_z(x, y, z) &= -x \sin(xz) + y \cos(yz)
\end{align*}
\]

\[
\nabla \mathbf{f}(\pi, 1, 0) = \langle \cos(\pi) - 0, 0, \pi \cos(\pi) + 0, 0 + \cos 0 \rangle = \langle -1, -\pi, 1 \rangle
\]

Tangent Plane: \(-1(x-\pi) - \pi(y-1) + 1(z-0) = 0\)

\[-x - \pi y + z = -2\pi\]

\[
\text{(b) Normal Line:} \quad \begin{cases} 
  x = \pi - t \\
  y = 1 - \pi t \\
  z = t
\end{cases} \quad \text{OR} \quad \langle \pi, 1, 0 \rangle + t(\langle -1, -\pi, 1 \rangle)
\]

\[
\text{(c) Angle between planes is angle between normal vectors} \\
\mathbf{n}_1 = \langle -1, -\pi, 1 \rangle \text{ and } \mathbf{n}_2 = \langle 0, 0, 1 \rangle
\]

\[
\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{||\mathbf{n}_1|| ||\mathbf{n}_2||} = \frac{1}{\sqrt{1 + \pi^2 + 1}} \quad \Rightarrow \quad \theta = \arccos \left( \frac{1}{\sqrt{1 + \pi^2}} \right)
\]
(15 points) Find the tangent line to the curve of intersection of the ellipsoid
\[ x^2 + \frac{y^2}{6} - \frac{z^2}{2} = \frac{1}{2} \]
and the plane \( x + 2y - 3z = 1 \) at the point \((1,3,2)\). You may express your answer as a set of parametric equations or as a vector equation.

\[ f(x, y, z) = x^2 + \frac{y^2}{6} - \frac{z^2}{2} \]
\[ f_x = 2x \]
\[ f_y = \frac{1}{3}y \]
\[ f_z = -z \]
\[ \nabla f(l, 3, 2) = \langle 2, 1, -2 \rangle \]

\[ g(x, y, z) = x + 2y - 3z \]
\[ g_x = 1 \]
\[ g_y = 2 \]
\[ g_z = -3 \]
\[ \nabla g(l, 3, 2) = \langle 1, 2, -3 \rangle \]

\[ \nabla f(l, 3, 2) \times \nabla g(l, 3, 2) = \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 1 & 2 & -3 \end{vmatrix} \]
\[ = (-3 + 4) \vec{i} - (-6 + 2) \vec{j} + (4 - 1) \vec{k} \]
\[ = \langle 1, 4, 3 \rangle \]

is parallel to the tangent line

Tangent Line:
\[ x = 1 + t \]
\[ y = 3 + 4t \]
\[ z = 2 + 3t \]

OR \( \vec{r}(t) = \langle 1, 3, 2 \rangle + t \langle 1, 4, 3 \rangle \)
(15 points) Identify all critical points of the given function. Then classify each critical point as a relative maximum, relative minimum, or saddle point.

\[ f(x, y) = \frac{1}{3} x^3 - 2x + x^2 + 2xy + y^2 \]

\[ f_x = x^2 - 2 + 2x + 2y = 0 \]

\[ f_y = 2x + 2y = 0 \implies y = -x \]

So \( x^2 - 2 = 0 \implies x = \sqrt{2}, -\sqrt{2} \)

\[ y = -\sqrt{2}, \sqrt{2} \]

Two critical points: \((\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})\)

\[ f_{xx} = 2x + 2, \quad f_{yy} = 2, \quad f_{xy} = 2 \]

\[ D(x, y) = (2x + 2)(2) - 2^2 = 4x + 4 - 4 = 4x \]

\[ D(\sqrt{2}, -\sqrt{2}) = 4\sqrt{2} > 0 \quad \text{and} \quad f_{xx}(\sqrt{2}, -\sqrt{2}) = 2\sqrt{2} + 2 > 0 \]

So relative minimum at \((\sqrt{2}, -\sqrt{2})\)

\[ D(-\sqrt{2}, \sqrt{2}) = -4\sqrt{2} < 0 \]

So saddle point at \((-\sqrt{2}, \sqrt{2})\)
(15 points) Recall that the volume of a right circular cylinder of radius $r$ and height $h$ is $V = \pi r^2 h$. Suppose that $r$ is measured to be 10 cm. with a maximum error of 0.02 cm. and that $h$ is measured to be 5 cm. with a maximum error of 0.03 cm. Use differentials to approximate the maximum error in the calculated value of the volume $V$.

Given $|dr| \leq 0.02$, $|dh| \leq 0.03$

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$dV = 2\pi rh \, dr + \pi r^2 \, dh$$

At $(r, h) = (10, 5)$:

$$dV = 100\pi \, dr + 100\pi \, dh$$

So $|dV| \leq 100\pi \cdot (0.02) + 100\pi \cdot (0.03)$

$$= 2\pi + 3\pi = 5\pi$$

So approximate maximum error in volume is $5\pi \, \text{cm}^3$. 