INTERPOLATION OF OPERATORS: RIESZ-THORIN, MARCINKIEWICZ

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If we notice that, for $1 < p < r < q < \infty$,

$$L_p \cap L_q \subseteq L_r \subseteq L_p + L_q$$

then, if we have a linear operator $T$ which is bounded on $L_p$ and on $L_q$, it’s natural to wonder whether or not $T$ must be bounded on $L_r$. The theorems of Riesz-Thorin and Marcinkiewicz will provide us with an answer to this question. In this project, we will discuss the interpolation theorems of Riesz-Thorin and Marcinkiewicz along with their proofs and several applications. Our first goal is to prove the Riesz-Thorin Theorem but we will need to prove several lemmas beforehand. The Riesz-Thorin Theorem pertains to complex valued functions so this is assumed for all lemmas along with the theorem.

Lemma 1.1 (Hadamard’s Three-Line Theorem). Suppose $f(z)$ is a bounded and continuous function on $0 \leq \Re z \leq 1$ and analytic in the interior. Denote

$$M_\theta = \sup_{y \in \mathbb{R}} |f(\theta + iy)|$$

Then $M_\theta \leq M_0^{1-\theta} M_1^\theta$ for $0 \leq \theta \leq 1$.

Proof. First, we may assume that $M_0, M_1 > 0$ else $f = 0$ and the proof is trivial.

Case 1: $M_0 = M_1 = 1$.

Since $f(z)$ is bounded on $0 \leq \Re z \leq 1$, there exists $k > 0$ so that $|f(z)| \leq k$ on $0 \leq \Re z \leq 1$. Let $\epsilon > 0$ and define

$$F_\epsilon = \frac{f(z)}{1 + \epsilon z}.$$ 

Further, let $\Omega = \{ z : 0 < \Re z < 1 \}$ and $R = \{ z = x + iy : x \in [0, 1], y \in [-\frac{k}{\epsilon}, \frac{k}{\epsilon}] \}$. Let $z = x + iy \in \delta \Omega$. Then either $x = 0$ or $x = 1$.

Well, if $x = 0$, then

$$|F_\epsilon(z)| = \frac{|f(z)|}{1 + \epsilon z} = \frac{|f(z)|}{1 + \epsilon iy} = \frac{|f(z)|}{\sqrt{1 + (\epsilon y)^2}} \leq |f(z)| \leq M_0 = 1.$$ 

Now, suppose $x = 1$. Then,

$$|F_\epsilon(z)| = \frac{|f(z)|}{1 + \epsilon z} = \frac{|f(z)|}{1 + \epsilon + iey} \leq \frac{|f(z)|}{1 + \epsilon} \leq |f(z)| \leq M_1 = 1.$$ 

Also,

$$|F_\epsilon(z)| = \frac{|f(z)|}{1 + \epsilon z} = \frac{|f(z)|}{1 + \epsilon x + iey} \leq \frac{|f(z)|}{3(1 + \epsilon x + iey)} = \frac{|f(z)|}{\epsilon |y|} \leq \frac{k}{\epsilon |y|}.$$ 

So, for $z \in \delta R - \overline{\Omega}$, we have

$$|F_\epsilon(x \pm i\frac{k}{\epsilon})| \leq \frac{|f(x \pm i\frac{k}{\epsilon})|}{\epsilon |\frac{k}{\epsilon}|} = \frac{|f(x \pm i\frac{k}{\epsilon})|}{k} \leq \frac{k}{k} = 1.$$ 

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So, for a fixed $\epsilon > 0$, $|F_\epsilon(z)| \leq 1$ on $\delta R$. So, by the maximum modulus principle, $\max_{z \in R} |F_\epsilon(z)| \leq 1$. Also for $z = x + iy \in \Omega - R$, we have $|y| > \frac{k}{\epsilon}$, so, by (1),

$$|F_\epsilon(z)| \leq \frac{k}{\epsilon |y|} \leq \frac{k}{\epsilon |\frac{x}{\epsilon}|} \leq 1.$$  

Hence $|F_\epsilon(z)| \leq 1$ for all $z \in \Omega$. Then we have that $|f(z)| \leq |1 + \epsilon z| \leq 1 + \epsilon |z|$. Since this is true for all $\epsilon > 0$, we have that $|f(z)| \leq 1$. So, for $0 \leq \theta \leq 1$, we have $M_\theta \leq 1 = M_1^{1-\theta} M_0^\theta$.

Case 2: $M_0, M_1 > 0$.

Let

$$G(z) = \frac{f(z)}{M_0^{1-Rz} M_1^{Rz}}.$$  

Since $M_0^{1-Rz}$ and $M_1^{Rz}$ are both entire functions, $G(z)$ is continuous on $0 \leq Rz \leq 1$ and analytic in the interior. Further, for $0 \leq Rz \leq 1$,

$$|G(z)| = \frac{|f(z)|}{M_0^{1-Rz} M_1^{Rz}} \leq \frac{k}{\min(1, M_0) \min(1, M_1)}.$$  

So $G(z)$ is bounded on $0 \leq Rz \leq 1$. Also,

$$\sup_{y \in \mathbb{R}} |G(iy)| = \sup_{y \in \mathbb{R}} \frac{|f(iy)|}{M_0} = \frac{M_0}{M_0} = 1$$

and

$$\sup_{y \in \mathbb{R}} |G(1 + iy)| = \sup_{y \in \mathbb{R}} \frac{|f(1 + iy)|}{M_1} = \frac{M_1}{M_1} = 1$$

Hence, by case 1, for $0 \leq \theta \leq 1$ we have

$$\sup_{y \in \mathbb{R}} |G(\theta + iy)| \leq 1$$

i.e.,

$$\sup_{y \in \mathbb{R}} \frac{|f(\theta + iy)|}{M_0^{1-\theta} M_1^{\theta}} \leq 1$$

Therefore $M_\theta = \sup_{y \in \mathbb{R}} |f(\theta + iy)| \leq M_0^{1-\theta} M_1^{\theta}$.  

\[ \square \]

Lemma 1.2. If $f \in L_p$ and $1 \leq p < \infty$ then

$$\|f\|_p = \sup_{\|h\|_{p'} = 1} |\langle f, g \rangle|$$

where $\langle f, g \rangle = \int_U f g d\mu$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $h$ is a simple function with compact support.

Proof. First we will establish the validity of the statement when the supremum is taken over all $g \in L_{p'}$ where $\|g\|_{p'} = 1$. If we let $g$ be such then

$$|\langle f, g \rangle| \leq \int_U |f g| d\mu$$

$$\leq \left( \int_U |f|^p d\mu \right)^{1/p} \left( \int_U |g|^{p'} d\mu \right)^{1/p'}\quad \text{Holder}$$

$$= \|f\|_p \|g\|_{p'} = \|f\|_p$$
So $\sup_{\|g\|_{p'}=1} \|\langle f, g \rangle\| \leq \|f\|_p$. Let

$$h(x) = \frac{e^{-i \arg f(x)} |f(x)|^{p-1}}{\|f\|_p^{p-1}}$$

Well,

$$\|h\|_{p'}^{p'} = \int_U \left| \frac{e^{-i \arg f(x)} |f(x)|^{p-1}}{\|f\|_p^{p-1}} \right|^{p'} d\mu$$

$$= \frac{1}{\|f\|_p^{p'(p-1)}} \int_U |f(x)|^{p'(p-1)} d\mu$$

$$= \frac{1}{\|f\|_p^p} \int_U |f(x)|^p d\mu$$

$$= 1$$

since $p = p'(p-1)$.

Also,

$$\langle f, h \rangle = \int_U f(x) \frac{e^{-i \arg f(x)} |f(x)|^{p-1}}{\|f\|_p^{p-1}} d\mu$$

$$= \int_U |f(x)| |f(x)|^{p-1} \frac{1}{\|f\|_p^{p-1}} d\mu$$

$$= \frac{1}{\|f\|_p^{p-1}} \int_U |f(x)|^p d\mu$$

$$= \|f\|_p$$

So we have that $\|f\|_p = \sup_{\|g\|_{p'}=1} |\langle f, g \rangle|$. Now we want to show

$$\sup_{\|h\|_{p'}=1} |\langle f, g \rangle| = \sup_{\|g\|_{p'}=1} |\langle f, g \rangle|$$

where $g \in L_{p'}$ and $h$ is simple with compact support. Clearly

$$\sup_{\|h\|_{p'}=1} |\langle f, g \rangle| \leq \sup_{\|g\|_{p'}=1} |\langle f, g \rangle|$$

since $h \in L_{p'}$ for all $h$. For the other direction, let $\epsilon > 0$. Then there exists $g_0 \in L_{p'}$ so that $\|g_0\| = 1$ and

$$\sup_{\|g\|_{p'}=1} |\langle f, g \rangle| \leq |\langle f, g_0 \rangle| + \frac{\epsilon}{2}$$
Since $g_0 \in L_{p'}$ there exists a simple function $h_0$ with compact support so that $\|g_0 - h_0\|_{p'} < \frac{\epsilon}{2\|f\|_{p' + 1}}$. Then

$$\sup_{\|g\|_{p'} = 1} |\langle f, g \rangle| \leq |\langle f, g_0 \rangle| + \frac{\epsilon}{2}$$

$$= |\langle f, h_0 \rangle + \langle f, g_0 - h_0 \rangle| + \frac{\epsilon}{2}$$

$$\leq |\langle f, h_0 \rangle| + |\langle f, g_0 - h_0 \rangle| + \frac{\epsilon}{2}$$

$$= |\langle f, h_0 \rangle| + \left| \int_U f(g_0 - h_0) d\mu \right| + \frac{\epsilon}{2}$$

$$\leq \sup_{\|h\|_{p'} = 1} |\langle f, h \rangle| + \|f\|_p \|g_0 - h_0\|_{p'} + \frac{\epsilon}{2} \tag{Holder}$$

$$< \sup_{\|h\|_{p'} = 1} |\langle f, h \rangle| + \epsilon$$

\[ \square \]

We are now ready to prove the Riesz-Thorin Interpolation Theorem.

**Theorem 1.3 (Riesz-Thorin).** Let $T$ be a linear operator and let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ where $p_0 \neq p_1$ and $q_0 \neq q_1$. Suppose $T : L_{p_0}(\mu) \rightarrow L_{q_0}(\nu)$ is bounded with norm $M_0$ and $T : L_{p_1}(\mu) \rightarrow L_{q_1}(\nu)$ is bounded with norm $M_1$. Let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ where $0 < \theta < 1$. Then $T : L_p(\mu) \rightarrow L_q(\nu)$ is bounded with norm $M \leq M_0^{1-\theta} M_1^{\theta}$.

**Proof.** Case 1: $1 \leq p_0, p_1, q_0, q_1 < \infty$

Let $f \in L_p$ where $\|f\|_p = 1$. We want to prove that $\|T(f)\|_q \leq M_0^{1-\theta} M_1^{\theta}$. By lemma 1.2, it suffices to show, for any simple function $g$ with finite support, $|\int_U T(f) g d\mu| \leq M_0^{1-\theta} M_1^{\theta}$.

Case 1: $f$ is simple with compact support.

Let $f = \sum_{j=1}^n a_j 1_{A_j}$ and let $g = \sum_{k=1}^m b_k 1_{B_k}$ where $a_j, b_j \in \mathbb{C}$ and $(A_n)$ and $(B_n)$ are both disjoint collections of subsets of $U$(not necessarily disjoint from each other) where $\mu(A_j) < \infty$ and $\nu(B_k) < \infty$ for all $j, k$. Let $\alpha(z) = \frac{1-z}{p_0} + \frac{z}{p_1}$ and let $\beta(z) = \frac{1-z}{q_0} + \frac{z}{q_1}$. For any $z = x + iy$ where $0 \leq x \leq 1$, define

$$f_z(t) = |f(t)|^{\frac{\alpha(z)}{\alpha(0)}} e^{i\arg f(t)}$$

and

$$g_z(t) = |g(t)|^{\frac{1-\beta(z)}{1-\beta(0)}} e^{i\arg g(t)}$$

Also, let

$$F(z) = \int_V T(f_z) g_z d\nu$$
Claim: $F(z)$ is analytic on $0 < \Re z < 1$.

\[
F(z) = \int_V T(f_z)g_zd\nu
= \int_V T \left( |f(t)|^{\alpha(z)/\alpha(\theta)} e^{i \arg f(t)} \right) |g(t)|^{1-\beta(z)\bar{\theta}} e^{i \arg g(t)} d\nu
= \int_V T \left( \sum_{j=1}^n |a_j|^{\alpha(z)/\alpha(\theta)} e^{i \arg a_j} 1_{A_j} \right) \left( \sum_{k=1}^m |b_k|^{1-\beta(z)\bar{\theta}} e^{i \arg b_k} 1_{B_k} \right) d\nu
= \sum_{j=1}^n \sum_{k=1}^m |a_j|^{\alpha(z)/\alpha(\theta)} |b_k|^{1-\beta(z)\bar{\theta}} e^{i(\arg a_j+\arg b_k)} \int_V T(1_{A_j}) 1_{B_k} d\nu
\]

If we recall how $\alpha$ and $\beta$ are defined then we see that the above is just a linear combination of exponential functions so we actually have that $F(z)$ is entire. From this, we also see that $F(z)$ is bounded on $0 \leq \Re z \leq 1$.

Claim: $|F(iy)| \leq M_0$ and $|F(1+iy)| \leq M_1$ for all $y \in \mathbb{R}$.

Well,

\[
|F(iy)| = \left| \int_V T(f_{iy})g_{iy}d\nu \right|
\leq \left( \int_V |T(f_{iy})|^{q_0} \right)^{1/q_0} \left( \int_V |g_{iy}|^{q_0'} \right)^{1/q_0'} \quad \text{Holder}
= \|T(f_{iy})\|_{q_0} \|g_{iy}\|_{q_0'}
\leq \|T\|_{p_0 \rightarrow q_0} \|f_{iy}\|_{p_0} \|g_{iy}\|_{q_0'}
= M_0 \|f_{iy}\|_{p_0} \|g_{iy}\|_{q_0'}
\]

since $T$ is bounded from $L_{p_0}$ to $L_{q_0}$ with norm $M_0$. Further,

\[
\|f_{iy}\|_{p_0}^p = \int_U |f|^{\alpha(y)/\alpha(\theta)} e^{i \arg f(t)} d\mu
= \int_U |f|^p d\mu
= 1
\]

where we get (*) because

\[
\left| f(z) \right|^{\alpha(y)/\alpha(\theta)} = \left| f(z) \right|^{\Re(\alpha(y)/\alpha(\theta))} \left| f(z) \right|^{i \arg(\alpha(y)/\alpha(\theta))} = \left| f(z) \right|^{\Re(\alpha(y)/\alpha(\theta))} \left| f(z) \right|^{\Im(\alpha(y)/\alpha(\theta))}
\]


and
\[
\Re\left(\frac{\alpha(iy)}{\alpha(\theta)}\right) = \Re\left[\left(\frac{1 - iy}{p_0} + \frac{iy}{p_1}\right)\left(\frac{1 - \theta}{p_0} + \frac{\theta}{p_1}\right)^{-1}\right]
\]
\[
= \Re\left[\left(\frac{p_1 - ip_1y + ip_0y}{p_0p_1}\right)p\right]
\] since \(\left(\frac{1 - \theta}{p_0} + \frac{\theta}{p_1}\right)^{-1} = \left(\frac{1}{p}\right)^{-1} = p\)
\[
= \frac{p_1p}{p_0p_1}
\]
\[
= \frac{p}{p_0}
\]
So \(\Re\left(\frac{\alpha(iy)}{\alpha(\theta)}\right)p_0 = \frac{p}{p_0} p_0 = p\). Similarly, \(\|g_{iy}\|_{q_0} = 1\) so we have that \(|F(iy)| \leq M_0\). Also,
\[
|F(1 + iy)| = \left|\int_V T(f_{1+iy})g_{1+iy}d\nu\right|
\]
\[
\leq \|T(f_{1+iy})\|_{q_1} \|g_{1+iy}\|_{q_1'} 
\] Holder
\[
\leq \|T\|_{p_1 \rightarrow q_1} \|f_{1+iy}\|_{p_1} \|g_{1+iy}\|_{q_1'}
\]
\[
= M_1 \|f_{1+iy}\|_{p_1} \|g_{1+iy}\|_{q_1'}
\] (2)

Further,
\[
\|g_{1+iy}\|_{q_1'} = \int_V |g(t)|^{\frac{\beta(1+iy)}{1-\beta(\theta)}}|^{q_1'}d\nu
\]
\[
= \int_V |g(t)|^{\Re\left(\frac{1-\beta(1+iy)}{1-\beta(\theta)}\right)p_1'}d\nu
\]
\[
= \int_V |g(t)|^{q_1'}d\nu
\] (*)
\[
= \|g\|_{q_1'}
\]
\[
= 1
\]
where we get (*) because
\[
\Re\left(\frac{1-\beta(1+iy)}{1-\beta(\theta)}\right) = \Re\left[\left(1 - \frac{1 - (1 + iy)}{q_0} - \frac{(1 + iy)}{q_1}\right)\left(1 - \frac{(1 - \theta)}{q_0} + \frac{\theta}{q_1}\right)^{-1}\right]
\]
\[
= (1 - \frac{1}{q_1})(\frac{1}{q'})^{-1}
\]
\[
= \frac{q'}{q_1'}
\]

Similarly we have that \(\|f_{1+iy}\|_{p_1} = 1\) so from (2) we get that \(|F(1 + iy)| \leq M_1\). Then by lemma 1, for \(0 \leq \theta \leq 1\) we have
\[
\sup_{y \in \mathbb{R}} |F(\theta + iy)| \leq M_0^{1-\theta} M_1^\theta
\]

Well,
\[
f_{\theta}(t) = |f(t)|e^{i\arg f(t)} = f(t)
\]
and similarly \( g_\theta(t) = g(t) \), so
\[
|F(\theta)| = \left| \int_V T(f) g d\nu \right|
\]

Then
\[
\left| \int_V T(f) g d\nu \right| \leq \sup_{y \in \mathbb{R}} |F(\theta + iy)| \leq M_1^{1-\theta} M_1^\theta
\]

Since this is true for any simple function \( g \) with compact support, we have then that
\[
M = \|T\|_{p \rightarrow q} \leq M_1^{1-\theta} M_1^\theta
\]

Case 2: \( f \in L_p \).
Since \( f \in L_p \) there exists a sequence of simple functions with compact support \( (f_n)_{n \geq 1} \) so that \( |f_n| \leq |f| \) and \( f_n \rightarrow f \) pointwise. Let \( E = \{x : |f(x)| > 1\} \), let \( g = f\chi_E \) and \( g_n = f_n\chi_E \). Further, let \( h = f - g \) and \( h_n = f_n - g_n \). Without loss of generality assume \( p_0 < p_1 \). Then
\[
\|g\|_{p_0}^{p_0} = \int_U |g|^{p_0} d\mu
\]
\[
= \int_U |f|^{p_0}\chi_E d\mu
\]
\[
\leq \int_U |f|^{p}\chi_E d\mu
\]
\[
\leq \int_U |f|^{p} d\mu
\]
\[
< \infty
\]

So \( g \in L_{p_0} \). Also,
\[
\|h\|_{p_1}^{p_1} = \int_U |f - f\chi_E|^{p_1} d\mu
\]
\[
= \int_U |f(1 - \chi_E)|^{p_1} d\mu
\]
\[
= \int_U |f|^{p_1}\chi_E d\mu
\]
\[
\leq \int_U |f|^{p}\chi_E d\mu
\]
\[
\leq \int_U |f|^{p} d\mu
\]
\[
< \infty
\]

So \( h \in L_{p_1} \). Since \( f_n \rightarrow f \) a.e., \( |f_n| \leq |f| \) and \( |f| \in L_p \), by the dominated convergence theorem we have \( \|f_n - f\|_p \rightarrow 0 \) and so \( \|f_n\|_p \rightarrow \|f\|_p \). Similarly, by the dominated convergence theorem we
also have that  \( \| g_n - g \|_{p_0} \to 0 \) and lastly, since
\[
|h_n| = |f_n - g_n| \\
= |f_n(1 - \chi E)| \\
\leq |f|(1 - \chi E) \\
= |f - f\chi E| \\
= |h|
\]
h_n \to h \text{ a.e. and } h \in L_{p_1} \text{ we have that } \| h_n - h \|_{p_1} \to 0. \text{ Since } T \text{ is bounded from } L_{p_0} \text{ to } L_{q_0} \text{ and from } L_{p_1} \text{ to } L_{q_1}, \text{ we then have that } \| T(g_n) - T(g) \|_{q_0} \to 0 \text{ and } \| T(h_n) - T(h) \|_{q_1} \to 0 \text{ and so }
T(g_n) \text{ converges to } T(g) \text{ in measure and } T(h_n) \text{ converges to } T(h) \text{ in measure. So there exists a subsequence } T(g_{n_k}) \text{ of } T(g_n) \text{ so that } T(g_{n_k}) \to T(g) \text{ a.e. and there exists a subsequence } T(h_{m_k}) \text{ of } T(h_m) \text{ so that } T(h_{m_k}) \to T(h) \text{ a.e. Then }
T(f_n) = T(g_n) + T(h_n) \to T(h) + T(g) = T(f)
so, by Fatou’s lemma, we have
\[
\| T(f) \|_q \leq \liminf_{n \to \infty} \| T(f_n) \|_q \leq \liminf_{n \to \infty} M_0^{1-\theta} M_1^{\theta} \| f_n \|_q \leq M_0^{1-\theta} M_1^{\theta} \| f \|_q
\]

Case 2: Some of \( p_0, p_1, q_0, q_1 \) are \( \infty \).
The proof of Case 1 works here simply by interchanging some integrals with supremums.

To see the significance of the Reisz-Thorin theorem, we will now look at several of its applications.
For the first one we are using the notation that
\[
f * g(x) = \int_{\mathbb{R}} f(y) g(x - y) dy
\]

**Theorem 1.4 (Young’s Inequality).** Suppose \( 1 \leq p, q, r \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \). If \( f \in L_p \) and \( g \in L_q \) then \( f * g \in L_r \) and
\[
\| f * g \|_r \leq \| f \|_p \| g \|_q
\]

**Proof.** Case 1: \( p = 1 \) and \( r = q \).
Then,
\[
\| f * g \|_r = \| f * g \|_q \\
= \| \int_{\mathbb{R}} f(y) g(\cdot - y) dy \|_q \\
\leq \int_{\mathbb{R}} \| f(y) g(\cdot - y) \|_q dy \quad \text{Minkowski} \\
= \int_{\mathbb{R}} \| f(y) \|_q \| g(\cdot - y) \|_q dy \\
= \| g(\cdot - y) \|_q \int_{\mathbb{R}} |f(y)| dy \\
= \| g \|_q \| f \|_1
\]
Case 2: \( p = \frac{q}{q-1} \) and \( r = \infty \).
Then,
\[
\|f \ast g\|_r = \|f \ast g\|_\infty \\
= \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} f(y)g(x-y)dy \\
\leq \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |f(y)g(x-y)|dy \\
\leq \sup_{x \in \mathbb{R}} \|f\|_p \|g\|_{p'} \\
= \|f\|_p \|g\|_q \quad \text{since } q = p'
\]

Case 3: General case.
Fix \( q \) so that \( 1 \leq q \leq \infty \) and fix \( g \in L_q \). Let \( T \) be the linear operator defined by \( T(f) = f \ast g \) Then, by case 1,
\[
\|T(f)\| \leq \|g\|_q \|f\|_1
\]
and, by case 2, we have
\[
\|T(f)\|_\infty \leq \|f\|_p \|g\|_{p'}
\]
In reference to the statement of the Riesz-Thorin theorem we have \( p_0 = 1, q_0 = q, p_1 = q', q_1 = \infty, M_0 = \|g\|_q, \) and \( M_1 = \|g\|_q \). So, by the Riesz-Thorin Theorem we have that
\[
\|T(f)\|_q \leq M_0^{1-t}M_1^t \|f\|_{p_t} = \|g\|_q \|f\|_{p_t}, \quad \forall f \in L_{p_t}
\]
where
\[
\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}; \quad 0 < t < 1
\]
And, in our case, this is
\[
\frac{1}{p_t} = 1 - t + \frac{t}{q}, \quad \frac{1}{q_t} = \frac{1-t}{q}
\]
We want \( q_t = r \). If this is the case then the above gives that \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \).

For the next application of the Riesz-Thorin theorem we need to first have a definition.

**Definition 1.5.** Let \( f \in L_2[0,1] \). The set \( B = \{E_k = e^{2\pi ikx} : k \in \mathbb{Z}\} \) is an orthonormal basis for \( L_2[0,1] \) so
\[
f = \sum_{k=-\infty}^{\infty} \langle f, E_k \rangle E_k
\]
Then the sequence \( \{\langle f, E_k \rangle\}_{k=-\infty}^{\infty} \) is the **Fourier Transform** of \( f \), denoted \( \hat{f} \).

Note:
\[
\langle f, E_k \rangle = \int_{0}^{1} f(x)e^{2\pi ikx}dx = \int_{0}^{1} f(x)e^{-2\pi ikx}dx
\]
Theorem 1.6 (Hausdorff-Young Theorem). Suppose $p$ and $q$ are conjugate exponents and $1 \leq p \leq 2$. If $f \in L_p[0, 1]$ then $\hat{f} \in \ell_q$ and $\|\hat{f}\|_q \leq \|f\|_p$.

Proof. First, a minor discussion is in order. The Riesz-Thorin theorem is about operators from $L_p(X, \mathcal{M}, \mu)$ to $L_q(Y, \mathcal{N}, \nu)$ so how can we talk about $\ell_q$ instead? Let $\nu$ be the counting measure on $Y = \mathbb{Z}$, i.e., $\nu(E) = \text{card}(E)$ for all $E \subseteq \mathbb{Z}$. In this sense then we have that $L_q = \ell_q$. Let $T$ be the linear operator defined by $T(f) = \hat{f}$. First, consider $T : L_1 \rightarrow \ell_\infty$. Let $f \in L_1$ and $k \in \mathbb{Z}$. Then

$$|\langle f, E_k \rangle| = \left| \int_0^1 f(x)e^{-2\pi ikx}dx \right| \leq \int_0^1 |f(x)||e^{-2\pi ikx}|dx = \int_0^1 |f(x)|dx = \|f\|_1$$

This is true for all $k \in \mathbb{Z}$ so

$$\|\{\langle f, E_k \rangle\}_{k \in \mathbb{Z}}\|_\infty = \sup_{k \in \mathbb{Z}} |\langle f, E_k \rangle| \leq \|f\|_1$$

So we have that $\|T(f)\|_\infty \leq M_0 \|f\|_1$ where $M_0 \leq 1$. Further, consider $T : L_2 \rightarrow \ell_2$. Let $f \in L_2$. Then

$$\left( \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 \right)^{1/2} = \left( \sum_{n \in \mathbb{Z}} |\langle f, E_k \rangle|^2 \right)^{1/2} = \|f\|_2$$

by Parseval. So we have that $\|T(f)\|_2 \leq M_1 \|f\|_2$ where $M_1 = 1$. By Riesz-Thorin, for $0 \leq \theta \leq 1$, $T : L_p \rightarrow \ell_q$ is bounded with norm $M \leq M_0^{-\theta}M_1^\theta \leq 1$ where

$$\frac{1}{p} = 1 - \theta + \frac{\theta}{2}, \quad \frac{1}{q} = \frac{\theta}{2}; \quad 0 \leq \theta \leq 1$$

i.e., where $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. 

The next topic of discussion is the Marcinkiewicz Interpolation Theorem along with its proof and an application. First we need to look at several definitions along with some lemmas. Also, Marcinkiewicz’s theorem is about real-valued $L_p$ spaces so this is assumed for the remainder of this paper.

Definition 2.1. If $f$ is a measurable function on $(X, \mathcal{M}, \mu)$, we define its distribution function $\lambda_f : (0, \infty) \rightarrow [0, \infty]$ by $\lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\})$.

Definition 2.2. If $f$ is a measurable function on $X$ and $0 < p \leq \infty$, we define

$$[f]_p = \left( \sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) \right)^{1/p}, \quad 0 < p < \infty; \quad [f]_\infty = \|f\|_\infty$$

and we define weak $L_p$ to be the set of all $f \in (X, \mathcal{M}, \mu)$ where $[f]_p < \infty$.

Note:

(1) $[f]_p \leq \|f\|_p$ so $L_p \subseteq \text{weak } L_p$.

(2) For $0 < p < \infty$, $[\cdot]_p$ is not a norm as it fails the triangle inequality.

Definition 2.3. If $U$ and $V$ are vector spaces, a map $T : U \rightarrow V$ is called sublinear if

$$|T(cf)(x)| = |c||T(f)(x)| \quad \text{for all } c \in \mathbb{R}; \quad \text{and} \quad |T(f + g)(x)| \leq |T(f)(x)| + |T(g)(x)|$$
**Definition 2.4.** Let $T$ be a map from some vector space $V$ of measurable functions on $(X, \mathcal{M}, \mu)$ to the space of all measurable functions on $(Y, \mathcal{N}, \nu)$. A sublinear map $T$ is said to be **strong type** $(p, q)$, for $1 \leq p, q \leq \infty$, if $L_p(\mu) \subseteq V$, $T$ maps $L_p(\mu)$ into $L_q(\nu)$ and there exists a constant $c > 0$ so that

$$\|T(f)\|_q \leq c\|f\|_p$$

for all $f \in L_p(\mu)$. A linear map $T$ is said to be **weak type** $(p, q)$, for $1 \leq p \leq \infty$, $1 \leq q < \infty$, if $L_p(\mu) \subseteq V$, $T$ maps $L_p(\mu)$ into weak $L_q$, and there exists a constant $c > 0$ such that

$$[T(f)]_q \leq c\|f\|_p$$

for all $f \in L_p(\mu)$.

**Lemma 2.5.** If $f \in L_p$, where $1 \leq p < \infty$, then

$$\lambda_f(\alpha) \leq \frac{\|f\|_p^p}{\alpha^p}, \quad \alpha > 0; \quad \text{and} \quad \|f\|_p^p = p \int_0^\infty t^{p-1} \lambda_f(t) dt$$

*Proof.* Well,

$$\|f\|_p^p = \int_U |f(x)|^p d\mu \geq \int_{\{x \in U : |f(x)| > \alpha\}} |f(x)|^p d\mu \geq \int_{\{x \in U : |f(x)| > \alpha\}} \alpha^p d\mu = \alpha^p \mu(\{x \in U : |f(x)| > \alpha\}) = \alpha^p \lambda_f(\alpha)$$

So, $\lambda_f(\alpha) \leq \frac{\|f\|_p^p}{\alpha^p}$. Let $A = \{(x, s) : |f(x)|^p > s\}$. Then,

$$\int_U |f(x)|^p d\mu = \int_U \int_0^{|f(x)|^p} ds d\mu = \int_{U \times [0, \infty)} 1_A(x, s) ds d\mu = \int_{U \times [0, \infty)} 1_A(x, s) d\mu ds \quad \text{Fubini} = \int_0^\infty \mu(\{x \in U : |f(x)|^p > s\}) ds = \int_0^\infty \mu(\{x \in U : |f(x)|^p > t^p\})pt^{p-1} dt \quad \text{where } t = \sqrt[p]{s} = p \int_0^\infty t^{p-1} \mu(\{x \in U : |f(x)| > t\}) dt = p \int_0^\infty t^{p-1} \lambda_f(t) dt \quad \square
Lemma 2.6. If $f \in L_p$, where $1 \leq p < \infty$, then

(a) \[
\int_{\{x : |f(x)| > \alpha\}} |f|^p d\mu = \lambda_f(\alpha)\alpha^p + p \int_0^\infty t^{p-1}\lambda_f(t)dt; \quad \text{and}
\]

(b) \[
\int_{\{x : |f(x)| \leq \alpha\}} |f|^p d\mu = -\lambda_f(\alpha)\alpha^p + p \int_0^\alpha t^{p-1}\lambda_f(t)dt
\]

Proof. (a) Let $A = \{(x, s) : |f(x)|^p > s\}$

\[
\int_{\{x : |f(x)| > \alpha\}} |f|^p d\mu = \int_{\{x : |f(x)| > \alpha\}} \int_0^\infty |f(x)|^p ds d\mu
\]

\[
= \int_{\{x : |f(x)| > \alpha\}} \int_0^{\alpha^p} ds d\mu + \int_{\{x : |f(x)| > \alpha\}} \int_{\alpha^p}^\infty |f(x)|^p ds d\mu
\]

\[
= \alpha^p \int_{\{x : |f(x)| > \alpha\}} d\mu + \int_{U \times [\alpha^p, \infty)} 1_A(x, s) ds d\mu
\]

\[
= \alpha^p \lambda_f(\alpha) + \int_{\alpha^p} \mu(\{x : |f(x)|^p > s\}) ds
\]

\[
= \alpha^p \lambda_f(\alpha) + \int_{\alpha^p} \mu(\{x : |f(x)|^p > t^p\}) pt^{p-1} dt \quad \text{where } s = t^p
\]

\[
= \alpha^p \lambda_f(\alpha) + p \int_0^\infty t^{p-1}\lambda_f(t)dt
\]

(b) \[
\int_{\{x : |f(x)| \leq \alpha\}} |f|^p d\mu = \int_U |f|^p d\mu - \int_{\{x : |f(x)| > \alpha\}} |f|^p d\mu
\]

\[
= p \int_0^\infty t^{p-1}\lambda_f(t)dt - \alpha^p \lambda_f(\alpha) - p \int_0^\alpha t^{p-1}\lambda_f(t)dt \quad \text{by Lemma 2.4 and (a)}
\]

\[
= -\lambda_f(\alpha)\alpha^p + p \int_0^\alpha t^{p-1}\lambda_f(t)dt
\]

We are now ready to state and prove The Marcenkiewicz Interpolation Theorem. For this, we will denote the set of all $\mu$-measurable functions on a vector space $U$ by $\mathfrak{M}(U, \mu)$.

Theorem 2.7 (Marcenkiewicz). Let $1 \leq p_0 < p_1 \leq \infty$ and assume that $T : \mathfrak{M}(U, \mu) \to \mathfrak{M}(V, \nu)$ is a sublinear operator of weak type $(p_0, p_0)$ and $(p_1, p_1)$. Then, for each $p_0 < p < p_1$, $T$ is of strong type $(p, p)$.

Proof. Let $f \in L_p$ and define

\[
g_\alpha(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq \alpha \\ 0 & \text{if } |f(x)| > \alpha \end{cases}
\]
and
\[ h_\alpha(x) = \begin{cases} 
0 & \text{if } |f(x)| \leq \alpha \\
 f(x) & \text{if } |f(x)| > \alpha 
\end{cases} \]

Claim: \( g_\alpha \in L_{p_1} \) and \( h_\alpha \in L_{p_0} \).

Well,
\[ \|g_\alpha\|_{p_1} = \int_U |g_\alpha|^{p_1} d\mu = \alpha^{p_1} \int_U \frac{|g_\alpha|}{\alpha}^{p_1} d\mu \leq \alpha^{p_1} \int_U \frac{|g_\alpha|^p}{\alpha} d\mu \leq \alpha^{p_1-p} \int_U |f|^p d\mu < \infty \]

Similarly,
\[ \|h_\alpha\|_{p_0} = \int_U |h_\alpha|^{p_0} d\mu = \alpha^{p_0} \int_U \frac{|h_\alpha|}{\alpha}^{p_0} d\mu \leq \alpha^{p_0} \int_U \frac{|h_\alpha|^p}{\alpha} d\mu \leq \alpha^{p_0-p} \int_U |f|^p d\mu < \infty \]

Then \( f(x) = g_\alpha(x) + h_\alpha(x) \) and, since \( T \) is sublinear,
\[ |T(f(x))| \leq |T(g_\alpha(x))| + |T(h_\alpha(x))| \]

Suppose \( x \in U \) and \( |T(f(x))| > \alpha \). Then
\[ |T(g_\alpha(x))| + |T(h_\alpha(x))| > \alpha \]

so either
\[ |T(g_\alpha(x))| > \frac{\alpha}{2} \quad \text{or} \quad |T(h_\alpha(x))| > \frac{\alpha}{2} \]

so
\[ \{x : |T(f(x))| > \alpha\} \subseteq \{x : |T(g_\alpha(x))| > \frac{\alpha}{2}\} \cup \{x : |T(h_\alpha(x))| > \frac{\alpha}{2}\} \]

and therefore
\[ \lambda_{Tf}(\alpha) \leq \lambda_{Tg_\alpha}(\frac{\alpha}{2}) + \lambda_{Th_\alpha}(\frac{\alpha}{2}) \]

Since \( T \) is weak \((p_1, p_1)\), there exists a constant \( c > 0 \) so that \( |T(g_\alpha)|_{p_1} \leq c\|g_\alpha\|_{p_1} \), i.e.,
\[ \left( \sup_{\alpha' > 0} \alpha'^{p_1} \lambda_{Tg_\alpha}(\alpha') \right)^{1/p} \leq c \left( \int_U |g|^{p_1} d\mu \right)^{1/p} \]

\[ \Rightarrow \left( \frac{\alpha}{2} \right)^{p_1} \lambda_{Tg_\alpha}\left(\frac{\alpha}{2}\right) \leq c_1 \int_U |g|^{p_1} d\mu \]

\[ \Rightarrow \lambda_{Tg_\alpha}\left(\frac{\alpha}{2}\right) \leq c_1 \left( \frac{2}{\alpha} \right)^{p_1} \int_U |g|^{p_1} d\mu \]

\[ = c_1 \left( \frac{2}{\alpha} \right)^{p_1} \int_{\{x : |f(x)| \leq \alpha\}} |f|^{p_1} d\mu \]

Similarly, since \( T \) is weak \((p_0, p_0)\),
\[ \lambda_{Th_\alpha}\left(\frac{\alpha}{2}\right) \leq c_2 \left( \frac{2}{\alpha} \right)^{p_1} \int_{\{x : |f(x)| > \alpha\}} |f|^{p_0} d\mu \]
Then
\[
\|T(f)\|_p^p = p \int_0^\infty \alpha^{p-1} \lambda_{Tf}(\alpha) d\alpha \\
\leq p \int_0^\infty \alpha^{p-1} \lambda_{Tg_\alpha}(\alpha) d\alpha + p \int_0^\infty \alpha^{p-1} \lambda_{Th_\alpha}(\alpha) d\alpha \\
\leq c_1 p 2^{p_1} \int_0^\infty \frac{\alpha^{p-1}}{\alpha^{p_1}} \int_{\{x:|f(x)| \leq \alpha\}} |f|^{p_1} d\mu d\alpha + c_2 p 2^{p_0} \int_0^\infty \frac{\alpha^{p-1}}{\alpha^{p_0}} \int_{\{x:|f(x)| > \alpha\}} |f|^{p_0} d\mu d\alpha \\
= c_1 p 2^{p_1} \int_0^\infty \alpha^{p-1-p_1} \left(-\lambda_f(\alpha) \alpha^{p_1} + p_1 \int_0^\alpha t^{p_1-1} \lambda_f(t) dt\right) d\alpha \\
+ c_2 p 2^{p_0} \int_0^\infty \alpha^{p-p_0-1} \left(\lambda_f(\alpha) \alpha^{p_0} + p_0 \int_\alpha^\infty t^{p_0-1} \lambda_f(t) dt\right) d\alpha \\
\leq c_1 p 1 p 2^{p_1} \int_0^\infty \alpha^{p-1-p_1} \int_0^\alpha t^{p_1-1} \lambda_f(t) dt d\alpha + c_2 p 0 2^{p_0} \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha \\
+ pp_0 c 2^{p_0} \int_0^\infty \alpha^{p-p_0-1} \int_\alpha^\infty t^{p_0-1} \lambda_f(t) dt d\alpha \\
\text{and therefore,} \\
\|T(f)\|_p^p = c_1 p 1 p 2^{p_1} \int_0^\infty \int_t^\infty \alpha^{p-1-p_1} t^{p_1-1} \lambda_f(t) dt d\alpha + c_2 p 0 2^{p_0} \|f\|_p^p \\
+ pp_0 c 2^{p_0} \int_0^\infty \int_0^t \alpha^{p-p_0-1} t^{p_0-1} \lambda_f(t) dt d\alpha \\
= c_2 p 0 2^{p_0} \|f\|_p^p + c_1 p 1 p 2^{p_1} \int_0^\infty t^{p_1-1} \lambda_f(t) \int_t^\infty \alpha^{p-p_1-1} d\alpha dt \\
+ pp_0 c 2^{p_0} \int_0^\infty t^{p_0-1} \lambda_f(t) \int_0^t \alpha^{p-p_0-1} d\alpha dt \\
= c_2 p 0 2^{p_0} \|f\|_p^p + c_1 p 1 p 2^{p_1} \int_0^\infty \frac{1}{p_1-p} t^{p_1-1} \lambda_f(t) dt + pp_0 c 2^{p_0} \int_0^\infty \frac{1}{p-p_0} t^{p_0-1} \lambda_f(t) dt \\
= c_3 \|f\|_p^p + c_4 \|f\|_p^p + c_5 \|f\|_p^p \\
= c_6 \|f\|_p^p \\
\text{The case when } p_1 = \infty \text{ is similar by taking} \\
g_\alpha(x) = \begin{cases} \\ f(x) & \text{if } |f(x)| \leq \frac{\alpha}{2c_1} \\ \frac{\alpha}{2c_1} \text{sign}(f(x)) & \text{if } |f(x)| > \frac{\alpha}{2c_1} \\ \end{cases} \\
\text{and} \\
h_\alpha(x) = f(x) - g_\alpha(x) \\
\text{where } c_1 \text{ is a positive constant so that } \|T(f)\|_\infty \leq c_1 \|f\|_\infty. \\
\]

The last topic of this paper is then to see an application of Marcinkiewicz's Theorem but first we must recall some things about the Hardy-Littlewood maximal function.
Definition 2.8. Let $f \in L_1$. Then we define the Hardy–Littlewood maximal function $Mf$ by

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)|dy$$

Recall: The Maximal Theorem states that for all $f \in L_1$,

$$\mu(\{x : Mf(x) > \alpha\}) \leq \frac{c}{\alpha} \int_{\mathbb{R}} |f(x)|dx$$

Theorem 2.9. If $f \in L_p(\mathbb{R})$, where $1 < p < \infty$, then there exists a constant $c > 0$ so that $\|Mf\|_p \leq c\|f\|_p$.

Proof. Let $f \in L_1$ and let $\alpha > 0$. Then, by the Maximal Theorem

$$\mu(\{x : Mf(x) > \alpha\}) \leq \frac{c}{\alpha} \int_{\mathbb{R}} |f(x)|dx$$

i.e.,

$$\alpha \lambda_{Mf}(\alpha) \leq c\|f\|_1$$

and so

$$[Mf]_1 \leq c\|f\|_1$$

Also, if $f \in L_{\infty}$ then

$$\|Mf\|_{\infty} = \text{ess sup}_{x \in \mathbb{R}} \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)|dy$$

$$\leq \text{ess sup}_{x \in \mathbb{R}} \sup_{r>0} \frac{1}{\mu(B_r(x))} \|f\|_{\infty} \mu(B_r(x))$$

$$= \|f\|_{\infty}$$

So, by Marcinkiewicz, for all $1 < p < \infty$, there exists $c_p > 0$ so that $\|Mf\|_p \leq c_p\|f\|_p$ for all $f \in L_p$.

Another application of Marcinkiewicz is in showing the Hilbert transform is bounded from $L_p$ to $L_p$ when $1 < p < \infty$. First, we will look at the definition.

Definition 2.10. For $f \in L_p$ where $1 < p < \infty$ the Hilbert transform of $f$, denoted $Hf$, is given by

$$Hf(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|x-y| \geq \epsilon} \frac{f(y)}{x-y}dy.$$  

The proof of the following theorem will not be given in this project, however it can be proven with Marcinkiewicz along with some basic properties of the Hilbert transform.

Theorem 2.11. For all $1 < p < \infty$, there exists a constant $C_p > 0$ such that

$$\|Hf\|_p \leq C_p\|f\|_p \quad \text{for all} \quad f \in L_p.$$  

While the interpolation theorems of Riesz-Thorin and Marcinkiewicz are interesting themselves, we also see they are very useful tools. They give us a way to prove certain statements which have possibly infinitely many cases by simply considering two of them.
References

