AN EXAMPLE OF A NON-SEPARABLE HILBERT SPACE

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Let \( C(\mathbb{R}) \) be the set of complex valued functions on \( \mathbb{R} \) and define \( \| \cdot \| : C(\mathbb{R}) \to [0, \infty] \) by

\[
\|f\|_0 = \left( \lim_{N \to \infty} \frac{1}{N} \int_{-N}^{N} |f(x)|^2 dx \right)^{1/2}
\]

and let \( \mathcal{H}_0 = \{ f \in C(\mathbb{R}) : \|f\|_0 < \infty \} \). Despite the notation, \( \| \cdot \|_0 \) is not a norm on \( \mathcal{H}_0 \). In fact, for any \( f \in L_2(\mathbb{R}) \) we have that \( \|f\|_0 = 0 \). We do, however, have that \( \| \cdot \|_0 \) is a semi-norm on \( \mathcal{H}_0 \) so, if we let \( I = \{ f \in \mathcal{H}_0 : \|f\|_0 = 0 \} \), then \( I \) is a closed subspace of \( \mathcal{H}_0 \) and so \( \mathcal{H}_1 = \mathcal{H}_0/I \) is a normed vector space with norm \( \| \cdot \| \) defined by

\[
\|f + I\| = \inf_{\varphi \in I} \|f - \varphi\|_0
\]

for all \( f + I \in \mathcal{H}_1 \). Note, by how we defined \( I \), that for any \( f \in \mathcal{H}_0 \) we have \( \|f + I\| = \|f\|_0 \). Now define \( \langle \cdot, \cdot \rangle : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C} \) by

\[
\langle f + I, g + I \rangle = \lim_{N \to \infty} \frac{1}{N} \int_{-N}^{N} f(x)\overline{g(x)} dx
\]

for all \( f + I, g + I \in \mathcal{H}_1 \) to make \( \mathcal{H}_1 \) an inner product space. Then let \( \mathcal{H} \) be the completion of \( \mathcal{H}_1 \) and so \( \mathcal{H} \) is a Hilbert space containing \( \mathcal{H}_1 \). (The reader may be curious as to how we take an existential completion such as this. If we let \( j : \mathcal{H}_1 \to \mathcal{H}_1^{**} \) be the natural map of \( \mathcal{H}_1 \) into its bidual, i.e., for \( x \in \mathcal{H}_1 \), \( j(x) : \mathcal{H}_1^* \to \mathbb{C} \) where \( j(x)(\phi) = \phi(x) \) for all \( \phi \in \mathcal{H}_1^* \), then \( j \) is a linear isometric embedding of \( \mathcal{H}_1 \) into \( \mathcal{H}_1^{**} \) and so we take the closure of \( \mathcal{H}_1 \) in \( \mathcal{H}_1^{**} \) to get \( \mathcal{H} \). Further, since \( \mathcal{H}_1^{**} \) is complete, we then have that \( \mathcal{H} \) is complete.) Now we have our Hilbert space \( \mathcal{H} \). Why is it non-separable? Consider \( \mathcal{B} = \{ \sin(\alpha x) : \alpha \in \mathbb{R} \} \). It is easy to check that \( \mathcal{B} \) is an uncountable (clearly) orthonormal set. First, if we suppose \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha \neq \beta \) then

\[
|\langle \sin(\alpha x), \sin(\beta x) \rangle| = \left| \lim_{N \to \infty} \frac{1}{N} \int_{-N}^{N} \sin(\alpha x)\sin(\beta x) dx \right|
\]

\[
= \left| \lim_{N \to \infty} \frac{1}{2N} \int_{-N}^{N} \cos((\alpha - \beta)x) - \cos((\alpha + \beta)x) dx \right|
\]

\[
= \left| \lim_{N \to \infty} \frac{1}{2N} \left. \frac{\sin((\alpha - \beta)x)}{\alpha - \beta} \right|_{-N}^{N} - \left. \frac{\sin((\alpha + \beta)x)}{\alpha + \beta} \right|_{-N}^{N} \right|
\]

\[
\leq \lim_{N \to \infty} \frac{1}{2N} \left( \frac{2}{\alpha - \beta} + \frac{2}{\alpha + \beta} \right)
\]

\[
= 0
\]

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Further, for $\alpha = \beta$ we have

\[
\langle \sin(\alpha x), \sin(\alpha x) \rangle = \lim_{N \to \infty} \frac{1}{N} \int_{-N}^{N} \sin^2(\alpha x) \, dx
\]

\[
= \lim_{N \to \infty} \frac{1}{2N} \int_{-N}^{N} 1 - \cos(2\alpha x) \, dx
\]

\[
= \lim_{N \to \infty} \left( 1 - \frac{\sin(2\alpha N)}{2\alpha N} \right)
\]

\[
= 1
\]

So, we have that $B$ is an uncountable orthonormal set in $\mathcal{H}$ and so $\mathcal{H}$ cannot have a countable basis, hence $\mathcal{H}$ is non-separable.