The Asymptotic Distribution of Symbols on Staircase Tableaux Diagonals

Amanda Lohss

September 15, 2016
Outline

- Definition of Staircase Tableaux
- Connections/Motivation
- Previous Results
- Results
- Open Problems
Staircase Tableaux (Corteel-Williams (2010))

**Definition**

A staircase tableau of size n is a Young diagram of shape \((n, n-1, \ldots, 1)\) such that:

1. The boxes are empty or contain an \(\alpha\), \(\beta\), \(\gamma\), or \(\delta\).

**Figure:** An example of a staircase tableau of size 7.
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2. Every box on the diagonal contains a symbol.

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3. All boxes in the same column and above an $\alpha$ or $\gamma$ are empty.

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2. Every box on the diagonal contains a symbol.
3. All boxes in the same column and above an $\alpha$ or $\gamma$ are empty.
4. All boxes in the same row and to the left of an $\beta$ or $\delta$ are empty.

*Figure:* An example of a staircase tableau of size 7.
The rows and columns in a staircase tableau are numbered from 1 through $n$, beginning with the box in the NW-corner and continuing south and east respectively.

**Figure:** A staircase tableau with weight $\alpha^2 \beta^3 \gamma^3 \delta^2$. 

$\begin{array}{cccc}
\alpha & \gamma & \delta & \alpha \\
\beta & \gamma & \delta & \\
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The weight of a staircase tableau is the product of all its symbols.

\[ \alpha^2 \beta^3 \gamma^3 \delta^2 \]
As proven by Corteel and Williams, summing over the weight of all staircase tableaux gives,

\[
\sum_{S \in S_n} \text{wt}(S) = \prod_{i=0}^{n-1} (\alpha + \beta + \delta + \gamma + i(\alpha + \gamma)(\beta + \delta)).
\]

and therefore the total number of staircase tableaux is \(4^n \cdot n!\).

Figure: A staircase tableau with weight \(\alpha^2 \beta^3 \gamma^3 \delta^2\).
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Connections

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- According to Yau, the ASEP is “the default stochastic model for transport phenomena.”

- Numerous other connections such as Askey-Wilson polynomials, tree–like tableaux, permutation tableaux, and permutations.
A Markov Chain with $n$ sites.

\[ \begin{array}{c}
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\end{array} \]
The ASEP

A Markov Chain with $n$ sites.

Transition Probabilities:

- $\bigcirc \cdot \cdot \cdot \bigcirc \cdot \cdot \cdot \bigcirc \cdot \cdot \cdot \bigcirc$ to
- $\bigcirc$ to $\bigcirc \cdot \cdot \cdot \bigcirc \cdot \cdot \cdot \bigcirc \cdot \cdot \cdot \bigcirc$: $\frac{\alpha}{n+1}$
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\[
\begin{align*}
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A \bullet \circ B \text{ to } A \circ \bullet B &: \frac{u}{n+1} \\
A \bullet \circ B \text{ to } A \circ \bullet B &: \frac{q}{n+1} \\
A \circ \bullet B \text{ to } A \bullet \circ B &: \frac{\beta}{n+1}
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\]
Connection with the ASEP

Type of a staircase tableaux:
- for each $\alpha$ or $\delta$ on diagonal.
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Filling rules for $u$’s and $q$’s:
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2. $u$’s in all boxes north of a $\alpha$ or $\delta$ and $q$’s in all boxes north of a $\beta$ or $\gamma$. 
The steady state probability that the ASEP is in state $\eta$ is:

$$\frac{\sum_{T \in \mathcal{T}} \text{wt}(T)}{\sum_{S \in S_n} \text{wt}(S)}.$$
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where $a := \alpha^{-1}$ and $b := \beta^{-1}$.
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Random $\alpha/\beta$-staircase tableaux:

$$\mathbb{P}(S_n,\alpha,\beta = S) = \frac{\text{wt}(S)}{\alpha^n \beta^n (a + b)^n}.$$
Because of the ASEP, the following random variables are interesting:

1. $A_n^k$, the number of $\alpha$’s along the $k^{th}$ diagonal.

2. $B_n^k$, the number of $\beta$’s along the $k^{th}$ diagonal.

3. $X_n^k$, the number of non-empty boxes along the $k^{th}$ diagonal.
Some Previous Results (Hitczenko-Janson)

Previous Result:

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\frac{A_n^1 - n/2}{\sqrt{n}} \xrightarrow{d} N(0, 1/12)
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Conjecture:

\(A_n^k\) and \(B_n^k\) are asymptotically Poisson \((k \geq 2)\).
Let $\text{Pois}(\lambda)$ be a Poisson random variable with parameter $\lambda$. Then as $n \to \infty$:

$$A_n^k \xrightarrow{d} \text{Pois} \left( \frac{1}{2} \right) \quad B_n^k \xrightarrow{d} \text{Pois} \left( \frac{1}{2} \right) \quad X_n^k \xrightarrow{d} \text{Pois} (1)$$
Outline of Proof

Theorem (Method of Factorial Moments)

If a sequence of random variables \( \{X_k\}_{k=1}^{n} \) is such that

\[
\lim_{n \to \infty} \mathbb{E}(X_k)_r \to \lambda^r, \quad r = 0, 1, \ldots
\]

then,

\[
X_n \overset{d}{\to} \text{Pois}(\lambda)
\]

For this calculation, one needs to calculate \( \mathbb{P}(\alpha_{j_1} \cap \alpha_{j_2} \cdots \cap \alpha_{j_r}) \).
Method: conditional probability and induction

\[ P(\alpha_{j_1} \cap \alpha_{j_2} \cdots \cap \alpha_{j_r}) = P(\alpha_{j_2} \cdots \cap \alpha_{j_r} | \alpha_{j_1}) \cdot P(\alpha_{j_1}). \]

For the second and third diagonal, \( P(\alpha_{j_2} \cdots \cap \alpha_{j_r} | \alpha_{j_1}) \) can be computed by considering all cases.
2nd and 3rd Diagonals (with Paweł Hitczenko)

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Kth Diagonal

\[ \alpha \alpha \beta \]
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\[ \begin{array}{cccc}
\alpha & \alpha & \alpha & \\
\beta & \alpha & & \\
\beta & & & \\
\beta & & & \\
\end{array} \]
Define a symbol to be $D$-connected if it is not on the first diagonal and one of the following two conditions hold:

1. The symbol lies on $D$ but is not on the $k$th diagonal.
2. There exists a symbol above or to the left that is $D$-connected or lies on $D$. 
D-connected symbols

Lemma

Properties of D-connected symbols:

1. Any symbol in the same column as a D-connected $\alpha$ or the same row as a D-connected $\beta$ is also D-connected.

2. There are at most $k - 2$ D-connected symbols.

3. Each D-connected symbol can be paired uniquely with an opposite symbol on the first diagonal.
Notice that the weight of the larger tableau is $\alpha^2 \beta^2 \times$ the weight of the smaller tableau.
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Define $\mathcal{A}_{k,h}$ to be the set of all possible arrangements of $h$ $D$-connected symbols in a tableau of size $k$.

Example:

$$\mathcal{A}_{4,2} = \{ \beta_1^2 \cap \beta_1^3, \beta_1^2 \cap \alpha_3^2, \beta_1^3 \cap \alpha_3^2, \beta_2^2 \cap \alpha_2^3, \beta_1^2 \cap \alpha_2^2, \alpha_2^3 \cap \alpha_3^2 \}$$

Note that $\mathcal{A}_{k,0} = \{ \emptyset \}$ for all $k \geq 1$. 
Lemma

There exists a bijection between tableaux of size $n$ with $\alpha_1$ and triples $(h, a, T)$ where $0 \leq h \leq k - 2$, $a \in A_{k,h}$ and $T$ is a tableaux of size $n - h - 2$
An example of the bijection

Figure: The bijection when $n = 3$ and $k = 3$. 
Proof: Surjectivity
Key Result

If $C_{k,h} := |A_{k,h}|$,

$$\sum_{S \in \mathcal{T}_{n,\alpha}} \text{wt}(S) = \sum_{h=0}^{k-2} C_{k,h} \alpha^{h+2} \beta^{h+1} \sum_{T \in \mathcal{T}_{n-h-2,\alpha,h}} \text{wt}(T).$$

Therefore,

$$\Pr_{n,\alpha,\beta}(\alpha_1^k, \alpha_2^k, \ldots, \alpha_j^k) = \sum_{h=0}^{k-2} C_{k,h} \frac{b}{(n+a+b-1)_{h+2}} \Pr_{n-h-2,\alpha,\beta}(\alpha_2^{h-2}, \ldots, \alpha_j^{h-2}).$$
The Distribution of Alpha Boxes

**Theorem**

Let $1 \leq j_1 < \ldots < j_r \leq n - k + 1$. If

$$j_l \leq j_{l+1} - k, \quad \forall l = 1, 2, \ldots, r - 1$$

then

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = \prod_{l=1}^{r} \frac{j_{r-l+1}}{n^2} + O\left(\frac{1}{n^{r+1}}\right).$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = O\left(\frac{1}{n^r}\right).$$
Open Problems

- What about if $k$ is not fixed?
- What about other regions?
- Asymptotic joint distribution of symbols on different diagonals?
Thank you!