Corners in Tree–Like Tableaux

Paweł Hitczenko & Amanda Lohss

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- Tree–Like Tableaux
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- Occupied Corners in Tree–Like Tableaux
- Diagonal Boxes in Symmetric Tree–Like Tableaux
Tree–Like Tableaux (Aval, Boussicault, Nadeau 2011)

Figure: A tree–like tableaux of size 13. There are $n!$ tableaux of size $n$.

Definition

A tree-like tableaux of size $n$ is a Ferrers diagrams of half-perimeter $n + 1$ such that,

1. The box in the first column and first row is pointed.
2. Either all boxes to the left of a pointed box is empty or all boxes above are empty.
3. Every row and every column contains at least one point.
Definition: The "special point" is the right-most point among those that occur at the bottom of a column.

Add a pointed column for each north step and a pointed row for each west step.

If the step is below the special point, add a ribbon.
There is a bijection between tree–like tableaux and permutations which “preserves trees”.
Tree-like tableaux provide a combinatorial formula for the partially asymmetric simple exclusion process (PASEP), an important particle model with applications in physics, biology and biochemistry.

Figure: An example of the PASEP as defined by a Markov chain of size 8.
Figure: A tree–like tableau and its associated state of the PASEP as represented by a Markov chain of size 12. Tree-like tableaux provide a combinatorial formula for the PASEP.
Conjecture: The number of corners in tree–like tableaux of size $n$ is $n! \times \frac{n+4}{6}$.

Conjecture: The number of corners in symmetric tree–like tableaux of size $2n + 1$ is $2^n \times n! \times \frac{4n+13}{12}$.
Permutation Tableaux

Figure: A permutation tableaux of size 12.

Definition

A permutation tableaux of size $n$ is a Ferrers diagram of half-perimeter $n$ such that

1. There is at least one 1 in every column.
2. There is no 0 with a 1 above it and a 1 to the left of it simultaneously.
The Bijection: (Aval, Boussicault, Nadeau 2011)

\[ c(T_n) = c(P_n) + |\{ P \in P_n : M_n(P) = S \}| = c(P_n) + (n - 1)! \]

**Theorem (Hitczenko, L.)**

For permutation tableaux of size \( n \),

\[ \mathbb{E}_n C_n = \frac{n + 4}{6} - \frac{1}{n}. \]
Figure: A symmetric tree–like tableaux of size 9. There are $2^n \cdot n!$ tableaux of size $n$. 
A type-B permutation tableaux of size $n$ is a shifted Ferrers diagram of half-perimeter $n$ such that,

1. The rules of permutation tableaux are satisfied.
2. If there is a 0 on the diagonal, it is a 0-row.

Figure: A type-B permutation tableaux of size 6.
The Bijection: (Aval, Boussicault, Nadeau 2011)

Figure: An example of the bijection between type–B permutation tableaux and symmetric tree–like tableaux.

1. Add a column and a root point then point unrestricted rows.
2. Replace all $0_R$’s with points (except on 0-rows).
3. Replace all non-diagonal $1_T$’s with points.
4. Make symmetric.
The Bijection

Key Relationship:

\[
c(T_{2n+1}^{sym}) = 2c(B_n) + 2|\{B \in B_n : M_n(B) = S\}| + |\{B \in B_n : M_1(B) = W\}| = c(B_n) + 2^n(n - 1)! + 2^{n-1}n!,
\]

Figure: Transformation of the Shape
If $U_{n-1}$ is the number of unrestricted rows, then there are $2^{U_{n-1}+1}$ extensions.

\[
\mathbb{P}(U_n = U_{n-1} + 1|F_{n-1}) = \frac{2}{2^{U_{n-1}+1}} = \frac{1}{2^{U_{n-1}}}.
\]

\[
\mathbb{P}(U_n = k|F_{n-1}) = \frac{1}{2^{U_{n-1}+1}} \left( \binom{U_{n-1}}{k-1} + \binom{U_{n-1}}{k-1} \right)
= \frac{1}{2^{U_{n-1}}} \binom{U_{n-1}}{k-1}.
\]

Therefore,

\[
\mathcal{L}(U_n|F_{n-1}) = 1 + \text{Bin}(U_{n-1}, 1/2).
\]
For all $B \in \mathcal{B}_{n-1}$,

$$
\mathbb{P}_n(B) = \frac{2^U_{n-1}(B)+1}{|\mathcal{B}_n|} = 2^U_{n-1}(B)+1 \frac{|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|} \mathbb{P}_{n-1}(B)
$$

Therefore, for any random variable $X$ on $\mathcal{B}_{n-1}$,

$$
\mathbb{E}_n(X) = \frac{2|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|} \mathbb{E}_{n-1}(2^U_{n-1}(B_{n-1}) X)
$$
Proposition (known but new proof: Hitczenko, L.)

For all $n \geq 0$, $|B_n| = 2^n n!$.

Proof can be deduced using the following:

$$\mathcal{L}(U_n|F_{n-1}) = 1 + \text{Bin}(U_{n-1}, 1/2).$$

$$\mathbb{E}_n(X) = \frac{2|B_{n-1}|}{|B_n|} \mathbb{E}_{n-1}(2^{U_{n-1}(B_{n-1})} X)$$

$$\mathbb{E}(a^{\text{Bin}(m)}) = \left(\frac{a + 1}{2}\right)$$
Theorem (Hitczenko, L.)

For type-B permutation tableaux of size $n$ we have

$$\mathbb{E}_n C_n = \frac{4n + 7}{24} - \frac{1}{2n}.$$ 

Proof.

$$\mathbb{E}_n \left( \sum_{k=1}^{n-1} I_{M_k=S, M_{k+1}=W} \right) = \sum_{k=1}^{n-1} \mathbb{E}_n \left( I_{M_k=S, M_{k+1}=W} \right)$$
The generating polynomial $P_n(x) := \sum_{T \in \mathcal{T}_n} x^{\text{oc}(T)}$ satisfies the following recurrence:

\[
\begin{cases}
    P'_n(x) = nP_{n-1}(x) + 2(1 - x)P'_{n-1}(x) \\
P_0(x) = 1
\end{cases}
\]

**Theorem (Hitczenko, L.)**

As $n \to \infty$, the limiting distribution of the number of occupied corners in a random tree–like tableau of size $n$ is $\text{Pois}(1)$. 
Proof.

Theorem (Hitczenko, L.)

Let $P_n(x) = \sum_{k=0}^{m} a_{n,k} x^k$ satisfy,

$$P_n'(x) = f_n(x)P_{n-1}(x) + g_n(x)P_{n-1}'(x)$$

where $g_n(1) = 0$, $g_n'(x) = g_n = o(f_n)$, $f_n'(x) = 0$ and

$$f_n \frac{P_{n-1}(1)}{P_n(1)} \to c > 0, \quad \text{as} \quad n \to \infty.$$ 

If a sequence of random variables $X_n$ is defined by

$$\mathbb{P}(X_n = k) = \frac{a_{n,k}}{P_n(1)} = \frac{a_{n,k}}{\sum_j a_{n,j}},$$

then

$$X_n \overset{d}{\to} \text{Pois}(c) \quad \text{as} \quad n \to \infty.$$
If $B(n, k)$ is the number of symmetric tree–like tableaux of size $2n + 1$ with $k$ diagonal cells, then $B_n(x) = \sum_{k=1}^{n+1} B(n, k)x^k$ satisfies the following recurrence,

$$
\begin{cases}
B_n(x) = nx(x + 1)B_{n-1}(x) + x(1 - x^2)B'_{n-1}(x), \\
B_0(x) = x.
\end{cases}
$$
Theorem (Hitczenko, L.)

Let $D_n$ be the number of diagonal boxes in a random symmetric tree–like tableau of size $2n + 1$. As $n \to \infty$,

$$
\frac{D_n - 3(n + 1)/4}{\sqrt{7(n + 1)/48}} \xrightarrow{d} N(0, 1).
$$

Proof. Notice that

$$
P(D_n = k) = \frac{B(n, k)}{\sum_{k \geq 0} B(n, k)} = \frac{B(n, k)}{B_n(1)},
$$

and then the conclusion will follow if the variance of $D_n \to \infty$ as $n \to \infty$ and $B_n(x)$ has non-positive real roots.
Proposition (Hitczenko, L.)

The variance of the number of diagonal cells in a random symmetric tree–like tableaux of size $2n + 1$ is,

$$\text{var}(D_n) = \frac{7(n + 1)}{48}.$$ 

Proof. Straightforward computation of

$$\text{var}(D_n) = \mathbb{E}(D_n)^2 - \mathbb{E}^2 D_n + \mathbb{E}D_n.$$
Proposition (Hitczenko, L.)

For all $n \geq 0$, the polynomial $B_n(x)$

a) has degree $n + 1$ with all coefficients non-negative, and

b) all roots real and in the interval $[-1, 0]$.

Proof. Induction and Rolle’s Theorem.

Recall:

\[
\begin{cases}
B_{n+1}(x) = (n + 1)x(x + 1)B_n(x) + x(1 - x^2)B'_n(x), \\
B_0(x) = x.
\end{cases}
\]

Key Step:

\[
B_{n+1}(x) = \frac{x(1 - x^2)}{K_n(x)} \frac{d}{dx} [K_n(x)B_n(x)].
\]
Future Directions

1. Asymptotic distribution of corners
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2. Type-B permutation tableaux
Future Directions

1. Asymptotic distribution of corners
2. Type-B permutation tableaux
3. Problems significant to PASEP