# SMALL STRONG SOLUTIONS FOR TIME-DEPENDENT MEAN FIELD GAMES WITH LOCAL COUPLING

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ABSTRACT. For mean field games with local coupling, existence results are typically for weak solutions rather than strong solutions. We identify conditions on the Hamiltonian and the coupling which allow us to prove the existence of small, locally unique, strong solutions over any finite time interval in the case of local coupling; these conditions place us in the case of superquadratic Hamiltonians. For the regularity of solutions, we find that at each time in the interior of the time interval, the Fourier coefficients of the solutions decay exponentially. The method of proof is inspired by the work of Duchon and Robert on vortex sheets in incompressible fluids.

**Petites solutions fortes pour jeux à champ moyens avec une dépendance temporelle et un couplage local**

Résumé: Pour les jeux à champ moyens avec couplage local, les résultats d'existence sont typiquement obtenus pour des solutions faibles plutôt que pour des solutions fortes. Nous identifions des conditions sur le Hamiltonien et sur le couplage qui nous permettent de démontrer l'existence d'une solution forte, petite et localement unique pour tout intervalle de temps fini dans le cas d'un couplage local; ces conditions nous placent dans une situation de Hamiltonien super-quadratique. Pour la régularité des solutions, nous trouvons que, pour chaque point dans l'intérieur de l'intervalle de temps, les coefficients de Fourier des solutions décroissent exponentiellement. La preuve est inspirée par les travaux de Duchon et Robert sur les nappes de tourbillons de fluides incompressibles.

## 1. Introduction

We study solutions of the Mean Field Games system in $n$ spatial dimensions,

$$-u_t - \Delta u + H(t,x,Du) = F(t,x,m), \qquad (t,x) \in [0,T] \times \mathbb{T}^n, \tag{1}$$

$$m_t - \Delta m - \operatorname{div}(mH_p(t,x,Du)) = 0, \qquad (t,x) \in [0,T] \times \mathbb{T}^n. \tag{2}$$

Here, $t$ is the temporal variable, taken from the finite interval $[0,T]$, for some fixed $T > 0$. We have chosen spatially periodic boundary conditions, and thus the spatial variable $x$ belongs to the $n$-dimensional torus, $\mathbb{T}^n$. The function $H$ is known as the Hamiltonian, with $H_p$ denoting $\frac{\partial}{\partial p} H(t,x,p)$, the function $F$ is known as the coupling, $u$ is a value function arising from a specific application, and $m$ is a probability distribution. The system (1), (2) is taken with the conditions

$$m(0,x) = m_0(x), \qquad u(T,x) = G(x, m(T,x)), \tag{3}$$

for a given payoff function, $G$. We will show that, in a suitably chosen function space, if $m_0$ is chosen sufficiently close to a uniform distribution, then the system (1), (2), (3) has a (locally) unique strong solution. We mention that the smallness condition (on the perturbation of $m$ from a uniform distribution) is independent of the size of the temporal horizon, $T$.

The theory of mean field games was introduced by Lasry and Lions, with the early papers on the subject being [7], [8], [9]; a recent survey is [6]. The topic of mean field games arises from game theory, in studying games with a large number of players. Approximating such systems with an infinite number of players, PDE models are arrived at, such as (1), (2) above. Existence results have been given, such as existence results for weak solutions [8], [12]. Some existence results assume that the coupling function, $F$, is a nonlocal smoothing operator, such as a regularization by convolution, and this smoothing allows for the proof of existence of strong solutions. In the case of local coupling, to quote [12], “in general, only the existence of weak solutions whose regularity remains mostly an open question can be proved.”

Strong solutions have been shown to exist in the case of local coupling by Gomes, Pimentel, and Sánchez-Morgado in both the cases of subquadratic Hamiltonian [5] and superquadratic Hamiltonian [4]. The present work is complementary to [4] and [5]. In the present work, we find the existence of strong solutions in the case of local coupling and a superquadratic Hamiltonian; the differences with the work of Gomes et al. are that our assumptions on the Hamiltonian and related functions are simpler, at the expense that we only find small solutions. Furthermore, another significant difference is that for Gomes et al., the consideration of superquadratic or subquadratic concerns the behavior at infinity, while our focus on small solutions necessitates being concerned instead with behavior near the origin. We note that the boundary condition at time $t = T$ used in [4], in which $u(T, \cdot)$ is given explicitly, is more straightforward to treat than the implicit condition in (3).

In the present work, we study mean field games by way of an analogy with fluid mechanics. In incompressible fluid mechanics, the vortex sheet is famously known to have an ill-posed initial value problem [2], [10], [13]. This can be seen to be related to the fact that the evolution equations for the vortex sheet are elliptic in space-time; for elliptic equations, one should solve boundary value problems instead of initial value problems. A boundary value problem in space-time can be formed by specifying data at an initial time and at a final time. Indeed, this is the approach taken by Duchon and Robert for proving the existence of some vortex sheet solutions for all time [3]; they specified data at time $t = 0$, in addition to specifiying zero data at $T = \infty$. In [11], the author and Milgrom have adapted the method of Duchon and Robert to finite time horizons, for both the vortex sheet and certain families of Boussinesq equations introduced by Bona, Chen, and Saut [1].

The Duchon-Robert methodology begins with the writing a Duhamel formula which integrates forward in time from time zero, and which integrates backwards in time from the final time. If Lipschitz estimates are satisfied by the nonlinear terms, then one is then able to use a contraction mapping argument to show that solutions exist. The function spaces used by Duchon and Robert use exponential weights, so that (in their case of an infinite time horizon), Fourier coefficients of the solutions decay exponentially, with the exponential decay rate increasing as time increases. In the present work, we follow these ideas, adapting the exponential weights to the case of a finite time horizon.

## 2. Formulation, functional setting, and assumptions

We define $\mathbb{P}$ to be the projection which removes the mean of a function on the torus; so, $\mathbb{P}f = f - \frac{1}{\mathrm{vol}(\mathbb{T}^n)} \int_{\mathbb{T}^n} f(x)\,dx$. Notice that the mean of $u$ plays no role on the right-hand sides of (1) and (2). We define $w = \mathbb{P}u$, and we find the following evolution equation for $w$ :

$$-w_t - \Delta w + \mathbb{P}H(t, x, Dw) = \mathbb{P}F(t, x, m).$$

Since $m$ is a probability measure at each time, its integral over the spatial domain will always be equal to one; notice that its integral is conserved by the evolution equation (2). The average value of $m$ is therefore equal to $\bar{m} = 1/\mathrm{vol}(\mathbb{T}^n)$. It will be helpful in what follows to subtract this from $m$; we define $\mu = m - \bar{m}$. We then have the following evolution equation for $\mu$ :

$$\mu_t - \Delta\mu - \mathrm{div}(\mu H_p(t, x, Dw)) - \bar{m}\mathrm{div}(H_p(t, x, Dw)) = 0. \tag{4}$$

This is taken with initial condition $\mu(0, \cdot) = \mu_0 := m_0 - \bar{m}$. We define $E(t, x, \mu)$ by $E(t, x, \mu) = \mathbb{P}F(t, x, \mu + \bar{m})$. We may then rewrite the evolution equation for $w$ :

$$-w_t - \Delta w + \mathbb{P}H(t, x, Dw) = E(t, x, \mu). \tag{5}$$

This is taken with the terminal condition $w(T, \cdot) = \mathbb{P}G(\cdot, m(T, \cdot)) = \mathbb{P}G(\cdot, \mu(T, \cdot) + \bar{m}) =: \tilde{G}(\mu(T, \cdot))$. Our system to be solved, then, is (4), (5).

We use the Duhamel formula for (4) (we note that we are solving for $\mu$ forward in time, beginning at $t = 0$). It is helpful to have some notation for the Duhamel integral; we define $I^+$ to be the operator given by

$$(I^+ f)(t, \cdot) = \int_0^t e^{\Delta(t-s)} f(s, \cdot)\ ds.$$

Since we will also be integrating backwards from time $T$, we will need the operator $I^-$, defined by

$$(I^- f)(t, \cdot) = \int_t^T e^{\Delta(s-t)} f(s, \cdot)\ ds,$$

in what follows. We then have the following Duhamel formula for $\mu$ :

$$\mu(t, \cdot) = e^{\Delta t}\mu_0 + (I^+ \mathrm{div}(\mu H_p(\cdot, \cdot, Dw)))(t, \cdot) + \bar{m}(I^+(\mathrm{div}(H_p(\cdot, \cdot, Dw))))(t, \cdot). \tag{6}$$

We introduce the further notation $I_T$ as a special case of $I^+$, so that $I_T f = (I^+ f)(T, \cdot)$. Evaluating (6) at time $t = T$, and introducing more notation, we have $\mu(T, \cdot) = A(\mu, w)$, with

$$A(\mu, w) = e^{\Delta T}\mu_0 + I_T(\mathrm{div}(\mu H_p(\cdot, \cdot, Dw))) + \bar{m} I_T(\mathrm{div}(H_p(\cdot, \cdot, Dw))). \tag{7}$$

For $w$, we write a Duhamel integral which integrates backward in time from $T$ :

$$w(t, \cdot) = e^{\Delta(T-t)} w(T, \cdot) - I^-(\mathbb{P}H(\cdot, \cdot, Dw))(t) + I^-(E(\cdot, \cdot, \mu))(t).$$

Using the boundary condition $w(T, \cdot) = \tilde{G}(\mu(T, \cdot)) = \tilde{G}(A(\mu, w))$, this becomes

$$w(t, \cdot) = e^{\Delta(T-t)}\tilde{G}(A(\mu, w)) - I^-(\mathbb{P}H(\cdot, \cdot, Dw))(t) + I^-(E(\cdot, \cdot, \mu))(t). \tag{8}$$

We assume that the projection of the coupling can be expressed as

$$E(t, x, \mu) = \tau\mu + \mathbb{P}U(t, x, \mu), \tag{9}$$

where $\tau$ is constant, and where $U$ satisfies a Lipschitz condition to be specified below in (15). [For example, if the coupling were given by $F(t, x, m) = m^3$, then we have $\mathbb{P}F(\mu) = \mathbb{P}(\mu + \bar{m})^3 = 3\bar{m}^2\mu + \mathbb{P}(3\bar{m}\mu^2 + \mu^3)$; we take $\tau = 3\bar{m}^2$ in this case and $U(\mu) = 3\bar{m}\mu^2 + \mu^3$.] Replacing $E$ in (8) with (9), we have our Duhamel formula for $w$ :

$$w(t, \cdot) = e^{\Delta(T-t)}\tilde{G}(A(\mu, w)) - I^-(\mathbb{P}H(\cdot, \cdot, Dw))(t) + I^-(\mathbb{P}U(, \cdot, \cdot, \mu))(t) + \tau I^-(\mu)(t). \tag{10}$$

We define a mapping, $\mathcal{T}$, to be $\mathcal{T}(\mu, w) = (\mathcal{T}_1, \mathcal{T}_2)$, with $\mathcal{T}_1$ and $\mathcal{T}_2$ given as follows:

$$\begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix} = \begin{pmatrix} e^{\Delta t}\mu_0 + I^+\mathrm{div}(\mu H_p(\cdot, \cdot, Dw))(t) + \bar{m} I^+\mathrm{div}(H_p(\cdot, \cdot, Dw))(t) \\ e^{\Delta(T-t)}\tilde{G}(A(\mu, w)) - I^-(\mathbb{P}H(\cdot, \cdot, Dw))(t) + I^-(\mathbb{P}U(\cdot, \cdot, \mu)) + \tau I^-(\mathcal{T}_1)(t) \end{pmatrix}.$$

Here, we have used the right-hand side of (6) for the definition of $\mathcal{T}_1$. We have also used the right-hand side of (10) for the definition of $\mathcal{T}_2$, with the final $\mu$ being substituted using (6).

### 2.1. Function spaces.

Let $\alpha \in \left[0, \frac{T}{2}\right)$ be given. Define $\beta : [0, T] \to [0, \alpha]$ by

$$\beta(s) = \begin{cases} 2\alpha s/T, & s \in [0, T/2], \\ 2\alpha - 2\alpha s/T, & s \in [T/2, T]. \end{cases}$$

For a function $f$ defined on $\mathbb{T}^n$, we denote the Fourier coefficients either by $\hat{f}(k)$ or $\mathcal{F}f(k)$, with $k \in \mathbb{Z}^n$.

For any $j \in \mathbb{N}$, we define $B^j$ to be the set of all functions $g : \mathbb{T}^n \to \mathbb{R}$, of zero mean, such that $|g|_j < \infty$, where

$$|g|_j = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^j \left|\hat{g}(k)\right|.$$

For any $j \in \mathbb{N}$, we define the space $\mathcal{B}^j_\alpha$ to be the set of all $f : [0,T] \times \mathbb{T}^n \to \mathbb{R}$, continuous in time with values in $B^j$, such that $\|f\|_{\mathcal{B}^j_\alpha} < \infty$, where

$$\|f\|_{\mathcal{B}^j_\alpha} = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \sup_{t\in[0,T]} e^{\beta(t)|k|}|k|^j \left|\hat{f}(t,k)\right|.$$

Similarly to the corresponding spaces in [3], there is the algebra property

$$\|fg\|_{\mathcal{B}^0_\alpha} \leq \|f\|_{\mathcal{B}^0_\alpha}\|g\|_{\mathcal{B}^0_\alpha}. \tag{11}$$

This algebra property follows from the inequality, for $k \in \mathbb{Z}^n \setminus \{0\}$,

$$\left|e^{\rho|k|}\widehat{fg}(k)\right| \leq \sum_j \left|e^{\rho|k-j|}\hat{f}(k-j)\right| \left|e^{\rho|j|}\hat{g}(j)\right|,$$

for any $\rho \geq 0$. Taken with (11), the product rule then implies the algebra property for all $j \in \mathbb{N}$ : for any $j \in \mathbb{N}$, there exists $c_j > 0$ such that if $f \in \mathcal{B}^j_\alpha$, $g \in \mathcal{B}^j_\alpha$, then $\|fg\|_{\mathcal{B}^j_\alpha} \leq c_j\|f\|_{\mathcal{B}^j_\alpha}\|g\|_{\mathcal{B}^j_\alpha}$.

We can show that the operators $I^+$ and $I^-$ are bounded linear operators from $\mathcal{B}^j_\alpha$ to $\mathcal{B}^{j+2}_\alpha$, for any $j \in \mathbb{N}$. The operator norms of each, denoted simply by $\|I^+\|$ and $\|I^-\|$, are bounded as follows:

$$\|I^+\| \leq \frac{2T}{T-2\alpha}, \qquad \|I^-\| \leq \frac{2T}{T-2\alpha};$$

these bounds are independent of $j$.

2.2. **Assumptions.** We make the following assumptions (which are primarily Lipschitz assumptions) on the Hamiltonian, $H$, on the coupling, $U$, and on the payoff function, $\tilde{G}$.
**(A1)** $\tilde{G}(0) = 0$, and $\tilde{G}$ is Lipschitz in a neighborhood of the origin in $B^2$; thus, there exists $c > 0$ such that for all $a_1$, $a_2$ sufficiently small in $B^2$, we have

$$|\tilde{G}(a_1) - \tilde{G}(a_2)|_2 \leq c|a_1 - a_2|_2. \tag{12}$$

We make similar assumptions for $H$, $H_p$, and for $E$, using the function spaces $\mathcal{B}^j_\alpha$. However, we need these to be superlinear:
**(A2)** $H(\cdot,\cdot,0) = H_p(\cdot,\cdot,0) = 0$. There exists a continuous function $\Phi_1 : \mathcal{B}^2_\alpha \times \mathcal{B}^2_\alpha \to \mathbb{R}$ such that as $(w_1, w_2) \to 0$, we have $\Phi_1(w_1, w_2) \to 0$, and such that

$$\|\mathbb{P}H_p(\cdot,\cdot,Dw_1) - \mathbb{P}H_p(\cdot,\cdot,Dw_2)\|_{\mathcal{B}^1_\alpha} \leq \Phi_1(w_1,w_2)\|Dw_1 - Dw_2\|_{\mathcal{B}^1_\alpha}. \tag{13}$$

(We abuse notation slightly here; since $H_p(\cdot,\cdot,Dw)$ is an $n$-vector, it is measured in the space $(\mathcal{B}^1_\alpha)^n$, but we do not denote the norm of this space differently than $\mathcal{B}^1_\alpha$.) There exists a continuous function $\Phi_2 : \mathcal{B}^2_\alpha \times \mathcal{B}^2_\alpha \to \mathbb{R}$ such that as $(w_1, w_2) \to 0$, we have $\Phi_2(w_1, w_2) \to 0$, and such that

$$\|\mathbb{P}H(\cdot,\cdot,Dw_1) - \mathbb{P}H(\cdot,\cdot,Dw_2)\|_{\mathcal{B}^0_\alpha} \leq \Phi_2(w_1,w_2)\|Dw_1 - Dw_2\|_{\mathcal{B}^1_\alpha}. \tag{14}$$

There exists $c > 0$ such that the mean of $H_p$ (which we may denote by $(I-\mathbb{P})H_p(\cdot,\cdot,Dw)$) satisfies

$$\sup_{t\in[0,T]} |(I-\mathbb{P})H_p(\cdot,\cdot,Dw)(t)| \leq c\|w\|_{\mathcal{B}^2_\alpha},$$

for all $w$ in a bounded subset of $\mathcal{B}^2_\alpha$.

We note that the conditions **(A2)** are all satisfied if $H$ is given by a superquadratic polynomial of $Dw$, such as $H(t,x,p_1,\ldots,p_n) = a(t,x)p_ip_jp_\ell$, for any $i,j,\ell \in \{1,\ldots,n\}$, with $a \in \mathcal{B}^2_\alpha$. Another example is $H(t,x,Dw) = a(t,x)|Dw|^4$.

**Remark 1.** *Although an assumption that $H$ is convex is traditional for mean field games, this property plays no role in the current proof, so we do not assume it.*

**(A3)** $U(\cdot,\cdot,0)=0$. There exists a continuous function $\Phi_3:\mathcal{B}^2_\alpha\times\mathcal{B}^2_\alpha\to\mathbb{R}$ such that as $(\mu_1,\mu_2)\to 0$, we have $\Phi_3(\mu_1,\mu_2)\to 0$, and such that

$$\|U(\cdot,\cdot,\mu_1)-U(\cdot,\cdot,\mu_2)\|_{\mathcal{B}^0_\alpha}\le \Phi_3(\mu_1,\mu_2)\|\mu_1-\mu_2\|_{\mathcal{B}^2_\alpha}. \tag{15}$$

Note that in each of the assumptions **(A1)**, **(A2)**, and **(A3)**, the mapping properties are implicit; for example, $U$ maps $\mathcal{B}^2_\alpha$ to $\mathcal{B}^0_\alpha$.

## 3. Existence and regularity of solutions

We will prove existence of a solution by the contraction mapping theorem; we will show that $\mathcal{T}$ maps some $X$ to itself with a contracting property. We let $\alpha\in(0,\frac{T}{2})$ be given, and let $\mu_0\in B^2$ be given. We let $X$ be the closed ball in $\mathcal{B}^2_\alpha\times\mathcal{B}^2_\alpha$, centered at $(a_0,b_0)=(e^{\Delta t}\mu_0, e^{\Delta(T-t)}\tilde{G}(e^{\Delta T}\mu_0)+\tau I^-(e^{\Delta\cdot}\mu_0)(t))$, with radius $r_*$ (the value of $r_*$ will be determined in what follows). Note that $(a_0,b_0)$ can be made small by taking $\mu_0$ to be small.

### 3.1. $\mathcal{T}$ maps $X$ to $X$.

We begin by determining that $\mathcal{T}$ maps $X$ to $X$; this will require placing conditions on $\mu_0$ and on $r_*$. We must first know that $\mathcal{T}$ maps $X$ to $\mathcal{B}^2_\alpha\times\mathcal{B}^2_\alpha$. This follows from the condition $\alpha\in(0,\frac{T}{2})$, the mapping properties of $I^+$ and $I^-$, and the assumptions of Section 2.2.

For the first component, $\mathcal{T}_1$, it is sufficient that the following inequality be satisfied:

$$\|I^+\mathrm{div}(\mu H_p(\cdot,\cdot,Dw))\|_{\mathcal{B}^2_\alpha}+\bar{m}\|I^+\mathrm{div}(H_p(\cdot,\cdot,Dw))\|_{\mathcal{B}^2_\alpha}\le\frac{r_*}{2}, \tag{16}$$

for any $(\mu,w)\in X$. We will show that we may choose $\mu_0$ and $r_*$ appropriately so that

$$\bar{m}\|I^+\mathrm{div}(H_p(\cdot,\cdot,Dw))\|_{\mathcal{B}^2_\alpha}\le\frac{r_*}{4}, \tag{17}$$

and we will omit most other details. To begin, we use the mapping properties of $I^+$ and the Lipschitz condition (13), with $w\in X$, to conclude

$$\begin{aligned}\|I^+\mathrm{div}(H_p(\cdot,\cdot,Dw))\|_{\mathcal{B}^2_\alpha}&\le\|H_p(\cdot,\cdot,Dw)\|_{\mathcal{B}^1_\alpha}\le\Phi_1(w,0)\|w\|_{\mathcal{B}^2_\alpha}\\ &\le\Phi_1(w,0)\left(\|a_0\|_{\mathcal{B}^2_\alpha}+\|b_0\|_{\mathcal{B}^2_\alpha}+r_*\right)\le\Phi\left(\|a_0\|_{\mathcal{B}^2_\alpha}+\|b_0\|_{\mathcal{B}^2_\alpha}+r_*\right)\left(\|a_0\|_{\mathcal{B}^2_\alpha}+\|b_0\|_{\mathcal{B}^2_\alpha}+r_*\right),\end{aligned} \tag{18}$$

where $\Phi(r)=\max_{\|w\|_{\mathcal{B}^2_\alpha}\le r}|\Phi_1(w,0)|$, for $r>0$. Then, if we choose $\mu_0$ and $r_*$ such that

$$\Phi(\|a_0\|_{\mathcal{B}^2_\alpha}+\|b_0\|_{\mathcal{B}^2_\alpha}+r_*)\le\frac{1}{8\bar{m}},\qquad \|a_0\|_{\mathcal{B}^2_\alpha}+\|b_0\|_{\mathcal{B}^2_\alpha}\le r_*,$$

then (17) is indeed satisfied. We note that for many other estimates of this section, the algebra properties described in Section 2.1 are also helpful.

For the second component, $\mathcal{T}_2$, it is sufficient that the following inequality be satisfied:

$$\begin{aligned}&|\tilde{G}A(\mu,w)-\tilde{G}(e^{\Delta T}\mu_0)|_2+\|I^-\mathbb{P}H(\cdot,\cdot,Dw)\|_{\mathcal{B}^2_\alpha}+\|I^-\mathbb{P}U(\cdot,\cdot,\mu)\|_{\mathcal{B}^2_\alpha}\\ &\qquad+\tau\|I^-I^+\mathrm{div}(\mu H_p(\cdot,\cdot,Dw))\|_{\mathcal{B}^2_\alpha}+\tau\bar{m}\|I^-I^+\mathrm{div}(H_p(\cdot,\cdot,Dw))\|_{\mathcal{B}^2_\alpha}\le\frac{r_*}{2}.\end{aligned}$$

Using the Lipschitz estimate (12) and the definition (7), we have

$$|\tilde{G}A(\mu,w)-\tilde{G}(e^{\Delta T}\mu_0)|_2\le|I_T\mathrm{div}(\mu H_p(\cdot,\cdot,Dw))|_2+\bar{m}|I_T\mathrm{div}(H_p(\cdot,\cdot,Dw))|_2.$$

For any $f\in\mathcal{B}^2_\alpha$, using the definition of $I_T$ and the fact that $\beta(T)=0$, we have

$$|I_Tf|_2=\sum_{k\in\mathbb{Z}^n\setminus\{0\}}e^{\beta(T)}|k|^2|\mathcal{F}((I^+f)(T))(k)|\le\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\sup_{t\in[0,T]}e^{\beta(t)}|k|^2|\mathcal{F}((I^+f)(t))(k)|=\|I^+f\|_{\mathcal{B}^2_\alpha}.$$

Combining the above, to conclude the demonstration that $\mathcal{T}$ maps $X$ to $X$, it is sufficient to show that

$$\begin{aligned}(19)\quad &\|I^{+}\mathrm{div}(\mu H_p(\cdot,\cdot,Dw))\|_{\mathcal{B}^2_\alpha}+\bar{m}\|I^{+}\mathrm{div}(H_p(\cdot,\cdot,Dw))\|_{\mathcal{B}^2_\alpha}+\|I^{-}\mathbb{P}H(\cdot,\cdot,Dw)\|_{\mathcal{B}^2_\alpha}\\ &+\|I^{-}\mathbb{P}U(\cdot,\cdot,\mu)\|_{\mathcal{B}^2_\alpha}+\tau\|I^{-}I^{+}\mathrm{div}(\mu H_p(\cdot,\cdot,Dw))\|_{\mathcal{B}^2_\alpha}+\tau\bar{m}\|I^{-}I^{+}\mathrm{div}(H_p(\cdot,\cdot,Dw))\|_{\mathcal{B}^2_\alpha}\le\frac{r_*}{2}.\end{aligned}$$

It is again clear from the mapping properties of $I^+$ and $I^-$, and from (13), (14), and (15), that $r_*$ and $\mu_0$ can be chosen appropriately so that (19) will be satisfied.

3.2. **Contraction estimate.** We can demonstrate that if $(\mu_1, w_1) \in X$ and if $(\mu_2, w_2) \in X$, then there exists $\lambda \in (0,1)$ such that

$$\|\mathcal{T}(\mu_1,w_1)-\mathcal{T}(\mu_2,w_2)\|_{\mathcal{B}^2_\alpha}\le\lambda\left(\|\mu_1-\mu_2\|_{\mathcal{B}^2_\alpha}+\|w_1-w_2\|_{\mathcal{B}^2_\alpha}\right).$$

This is accomplished very similarly to the estimates of Section 3.1, and is true for sufficiently small $\mu_0$ and $r_*$. We omit the details.

3.3. **The main theorem.** We are ready to state and prove our main theorem.

**Theorem 1.** *Let $T > 0$ and $\alpha \in (0, \frac{T}{2})$ be given. Let* **(A1)***,* **(A2)***, and* **(A3)** *be satisfied. There exists $\delta > 0$ such that for all $m_0$ such that $(m_0 - \bar{m}) \in B^2$ with $|m_0 - \bar{m}|_2 < \delta$, there exists $u$ such that $\mathbb{P}u \in \mathcal{B}^2_\alpha$, and there exists $m$ such that $(m - \bar{m}) \in \mathcal{B}^2_\alpha$, with $u$ and $m$ being a solution of (1), (2), (3), and such that $m$ is a probability measure. Furthermore, for each $t \in (0,T)$, the Fourier coefficients of $u(t,\cdot)$ and $m(t,\cdot)$ decay exponentially.*

**Remark 2.** *The value of $\delta$ need not go to zero as $T$ increases; indeed, it is possible to run a version of this argument with $T = \infty$, as evidenced by the original Duchon-Robert argument* [3]*.*

**Proof of Theorem 1:** The contraction mapping theorem gives the existence of $\mu$, $w$ solving (4), (5). We let $m = \mu + \bar{m}$. From (3), we see that the mean of $u(T,\cdot)$ should equal the mean of $G(\cdot, m(T,\cdot))$. The mean of $u$ can then be found by integrating (1) in space and in time, starting from time $T$. We conclude that $u$ and $m$ solve (1), (2), (3). That the Fourier coefficients decay exponentially at each time $t \in (0,T)$ follows from the fact that $\beta(t) > 0$ for $t \in (0,T)$ and from the inequality $\sum_{k\in\mathbb{Z}^n\setminus\{0\}} e^{\beta(t)|k|}|\hat{f}(t,k)| \le \|f\|_{\mathcal{B}^0_\alpha}$, for any $f \in \mathcal{B}^0_\alpha$. ■

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