CONVERGENCE OF A BOUNDARY INTEGRAL METHOD FOR 3D INTERFACIAL DARCY FLOW WITH SURFACE TENSION

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ABSTRACT. We study convergence of a boundary integral method for 3D interfacial flow with surface tension when the fluid velocity is given by Darcy's Law. The method is closely related to a previous method developed and implemented by Ambrose, Siegel, and Tlupova, in which one of the main ideas is the use of an isothermal parameterization of the free surface. We prove convergence by proving consistency and stability, and the main challenge is to demonstrate energy estimates for the growth of errors. These estimates follow the general lines of estimates for continuous problems made by Ambrose and Masmoudi, in which there are good estimates available for the curvature of the free surface. To use this framework, we consider the curvature and the position of the free surface each to be evolving, rather than attempting to determine one of these from the other. We introduce a novel substitution which allows the needed estimates to close.

1. INTRODUCTION

Boundary integral methods are a family of commonly used methods for computing solutions of initial value problems for fluid interface problems, and there are two main reasons for their popularity. One is that when such methods apply, by reducing the computation to the boundary, the dimension of the problem is reduced; another reason is that boundary integral methods tend to be very accurate. For interfacial flow problems with surface tension, one disadvantage is that boundary integral methods can be very stiff. For example, for interfacial flow with surface tension with the fluid velocities given by Darcy's Law, a fully explicit method would face a third-order timestepping constraint, and if the velocities were instead given by the incompressible Euler equations, there would be a 3/2-order timestepping constraint. For more information on boundary integral methods for interfacial fluid flow, we refer the interested reader to the recent survey by Baker [9].

A breakthrough for boundary integral methods for interfacial flow with surface tension for two-dimensional fluids was made by Hou, Lowengrub, and Shelley (HLS) [17], [18]. HLS removed the stiffness from the problem by reformulating the problem using a geometric description of the free surface rather than using Cartesian variables, by using a convenient parameterization of the free surface, and by writing the evolution equations with a small-scale decomposition (SSD), extracting the most singular terms from singular integrals such as the Birkhoff-Rott integral. These choices made the evolution equations semilinear, which allows straightforward use of an implicit-explicit timestepping scheme [8], removing the strong stiffness constraint and leaving at worst a first-order timestepping constraint. The HLS work was subsequently extended by Nie to axisymmetric flow in 3D [21]. For doubly periodic interfaces in three-dimensional flows, Ambrose and Siegel introduced a non-stiff method for a model problem for interfacial Darcy flow with surface tension [6], and Ambrose, Siegel, and Tlupova subsequently treated the full interfacial Darcy problem with surface tension [7]

The method of [7] builds upon both the HLS numerical works and the analytical works of Ambrose and Masmoudi [2], [4], [5]. The work of Ambrose and Masmoudi gives well-posedness proofs for different problems in 3D interfacial fluid flow, again using a convenient parameterization of the free surface and making estimates for the mean curvature of the interface. The numerical method of [7]

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follows these ideas, using an isothermal parameterization for the free surface (see Section 2 below for more detail on this choice of parameterization), finding an SSD for the problem, and using an implicit-explicit timestepping scheme, thus removing the stiffness. One additional difficulty for the 3D problem which was not present in 2D is that a fast method for computing the velocity integral was needed, and a method based on Ewald summation was therefore developed in [7].

For the HLS method and other boundary integral methods for interfacial fluid flow, convergence of the methods has been established by several authors. Beale, Hou, and Lowengrub proved convergence of a boundary integral method for the 2D water wave without surface tension [12]; the water wave is the problem with a single fluid bounded above by a free surface, with the fluid velocity given by the Euler equations. Extensions of this to 3D are [11] and [20]. For 2D flow with surface tension, Ceniceros and Hou proved the convergence of the HLS method, in both the Darcy and Euler cases [15]. In these convergence proofs, the most important step is always to establish estimates for the growth of errors when proving stability. The estimates of these papers, generally speaking, built upon estimates for solutions of the linearized equations of motion proved by some of the same authors [13], [19].

While such estimates for the linearized motion of interfaces in 3D fluids with surface tension have been established [20], we are unaware of any proof in the literature of convergence of a boundary integral method for 3D interfacial flow in the presence of surface tension. This is the subject of the present work. As in [15], we study a semicontinuous problem, making discretizations in space but leaving time as continuous. We study the same problem as in [7], and consider the method of the current work to be a version of the method of [7]. We prove convergence of the method we develop by establishing consistency and stability.

As we have mentioned above, the most important step of the convergence proof is the proof of energy estimates in the stability analysis. The estimates we establish are thus related to the estimates of Ambrose and Masmoudi, as in the papers [2], [3], [4], [5]. For 3D interfacial flow problems, Ambrose and Masmoudi were able to establish estimates for the growth of the mean curvature of the interface, and then use these estimates to establish well-posedness of the initial value problems. In the current work, we follow this general framework, but there are of course additional challenges in the spatially discrete setting.

One main challenge is that in the continuous setting, the regularity of the free surface was able to be inferred from the regularity of the mean curvature: if we know that the surface is in the space H^s and the curvature is in the space H^s, for example, then we may infer that the surface is in fact in the space H^{s+2}. In the spatially discrete setting, with exact relationships between quantities either not holding or being more complicated, we are not able to make the corresponding inference. To deal with this difficulty, we introduce a novel substitution, replacing the free surface in the evolution equations by using a formula involving the curvature, before discretizing (see Section 5 for details). This substitution allows the energy estimates to close, yielding stability of the numerical method.

In making the energy estimates, we use in a fundamental way that the Darcy flow problem is of a parabolic nature. With a positive value of the surface tension parameter, there is a gain of $3/2$ of a spatial derivative at positive times [3]. (In the zero surface tension case, if a stability condition is satisfied, the problem is still a forward parabolic problem, but the smoothing effect is instead $1/2$ of a spatial derivative [25].) In future work, incompressible Euler problems (such as the Kelvin-Helmholtz or Rayleigh-Taylor problems) will be treated, and this parabolic effect will not be available. For these problems, the energy estimates will then need to be made somewhat more carefully.

The plan of the paper is as follows: in Section 2, we discuss the governing equations for 3D interfacial flow with surface tension, and give a boundary integral formulation related to the works [4] and [7]. In Section 3, we continue to develop the boundary integral formulation by working through the SSD. In Section 4 we develop evolution equations for the mean curvature of the interface and related quantities. We then are able to discretize the problem, and we give the numerical method in Section 5. The main theorem of the paper, Theorem 5.1, is stated at the end of Section 5. We prove consistency of this method in Section 6. In Section 7, we discuss bounds for discretized versions of some integral operators; these are of use in Section 8, in which stability, and ultimately convergence, are proved.

2. The problem and its boundary integral formulation

In this section we describe the 3D Darcy flow problem for which our numerical method is designed and represent it using a boundary integral formulation. The model problem is similar to that of [6, 7], and hence here we skip some of the details on formulating the governing equations.

We consider the flow of two immiscible, incompressible fluids that are separated by a free interface in a three-dimensional porous media, in which case the velocities of the fluids are determined by Darcy's law. In this paper, we use bold face letters to denote vector variables. Suppose that the interface is parametrized by the spatial variable $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) = (\alpha, \beta)$. Under the boundary integral formulation, the state of the system at time t is specified by the interface $\mathbf{X}(\boldsymbol{\alpha}, t) = (x(\boldsymbol{\alpha}, t), y(\boldsymbol{\alpha}, t), z(\boldsymbol{\alpha}, t))$. Throughout this paper we often suppress the time variable t and sometimes even the spatial variable $\boldsymbol{\alpha}$ as long as the involved terms do not lack clarity. For example, we often use $\mathbf{X}$ and $\mathbf{X}'$ to denote $\mathbf{X}(\boldsymbol{\alpha}, t)$ and $\mathbf{X}(\boldsymbol{\alpha}', t)$ respectively.

The flow is assumed to be of infinite depth in the z direction and 2π-periodic in both of the horizontal x and y directions, that is, $\mathbf{X}(\boldsymbol{\alpha} + (2k_1\pi, 2k_2\pi)) = \mathbf{X}(\boldsymbol{\alpha}) + (2k_1\pi, 2k_2\pi, 0)$, where k_1, k_2 are arbitrary integers. We denote the unit tangent and normal vectors by

$$\mathbf{t}_1 = \frac{\mathbf{X}_\alpha}{|\mathbf{X}_\alpha|}, \quad \mathbf{t}_2 = \frac{\mathbf{X}_\beta}{|\mathbf{X}_\beta|}, \quad \mathbf{n} = \mathbf{t}_1 \times \mathbf{t}_2, \tag{2.1}$$

respectively. The fundamental forms of the interface are defined as follows:

$$\begin{aligned} E &= \mathbf{X}_\alpha \cdot \mathbf{X}_\alpha, \quad F = \mathbf{X}_\alpha \cdot \mathbf{X}_\beta, \quad G = \mathbf{X}_\beta \cdot \mathbf{X}_\beta, \\ L &= -\mathbf{X}_\alpha \cdot \mathbf{n}_\alpha, \quad M = -\mathbf{X}_\alpha \cdot \mathbf{n}_\beta, \quad N = -\mathbf{X}_\beta \cdot \mathbf{n}_\beta. \end{aligned} \tag{2.2}$$

Following [2, 4, 6, 7], we choose an isothermal parametrization, that is, $E = G$, $F = 0$ (this requires a certain symmetry for the free surface; we discuss this more below). Motion of the fluids is driven by gravity and surface tension and a prescribed far-field pressure gradient, which produces a constant fluid velocity $V_\infty \hat{\mathbf{k}}$ as $z \to \pm\infty$, where $\hat{\mathbf{k}}$ is the unit vector in the z-direction. In this paper we assume $V_\infty = 0$ and use $\mathbf{V} = \mathbf{V}(\boldsymbol{\alpha}, t)$ to denote the velocity of the interface. Decomposing $\mathbf{V}$ in normal and tangential directions, we get

$$\frac{d\mathbf{X}}{dt} = \mathbf{V} = U\mathbf{n} + V_1\mathbf{X}_\alpha + V_2\mathbf{X}_\beta, \tag{2.3}$$

where

$$U = \mathbf{V} \cdot \mathbf{n}, \quad V_1 = \frac{\mathbf{V} \cdot \mathbf{t}_1}{\sqrt{E}}, \quad V_2 = \frac{\mathbf{V} \cdot \mathbf{t}_2}{\sqrt{E}}. \tag{2.4}$$

The normal component U can be calculated from

$$U = \mathbf{W} \cdot \mathbf{n}, \tag{2.5}$$

where $\mathbf{W}$ is the velocity of the interface under a Lagrangian frame. It is given by the Biot-Savart formula

$$\mathbf{W} = -\frac{1}{4\pi} PV \int \boldsymbol{\eta}' \times \frac{\mathbf{X} - \mathbf{X}'}{|\mathbf{X} - \mathbf{X}'|^3} d\boldsymbol{\alpha}', \tag{2.6}$$

where $\boldsymbol{\eta} = \mu_\alpha \mathbf{X}_\beta - \mu_\beta \mathbf{X}_\alpha$ is the unnormalized vortex sheet strength and μ is the dipole strength. We will also use the notations $r_1 = \mu_\alpha$, $r_2 = \mu_\beta$ and hence $\boldsymbol{\eta} = r_1\mathbf{X}_\beta - r_2\mathbf{X}_\alpha$. Here in this paper we assume that the two fluids are of the same density and the same viscosity; these assumptions allow us to focus on the motion due to surface tension, which is the highest-order of these effects. Therefore, as shown in [6], we have

$$r_1 = B\kappa_\alpha, \quad r_2 = B\kappa_\beta, \tag{2.7}$$

where κ is the mean curvature and B is the positive coefficient of surface tension. Because of the isothermal parametrization, we have

$$\Delta\mathbf{X} = 2\kappa\mathbf{N}, \tag{2.8}$$

where $\mathbf{N} = \mathbf{X}_\alpha \times \mathbf{X}_\beta$. The mean curvature κ can be calculated from

$$\kappa = \Delta\mathbf{X} \cdot \frac{\mathbf{n}}{2E} = \frac{L+N}{2E}. \tag{2.9}$$

Let $G(\mathbf{x}) = (4\pi r)^{-1}$, where $r = |\mathbf{x}|$. Then (2.6) can be written as

$$\mathbf{W} = PV \int \boldsymbol{\eta}' \times \nabla_{\mathbf{X}} G(\mathbf{X}(\boldsymbol{\alpha}) - \mathbf{X}(\boldsymbol{\alpha}'))d\boldsymbol{\alpha}'. \tag{2.10}$$

The tangential velocities V_1, V_2 are calculated from the isothermal assumption $E = G$, $F = 0$ which is maintained by imposing

$$E_t = G_t, \quad F_t = 0. \tag{2.11}$$

Remark 2.1. *To use this form of the isothermal parameterization, we are assuming a certain symmetry on the solutions. If this symmetry is initially present, it will be maintained at positive times. Specifically, we are assuming that we are able to parameterize the initial surface so that for all α and β, we have $E(\alpha,\beta,0) = G(\alpha,\beta,0)$. Such a parameterization is not always possible when insisting that the parameters (α,β) is from the set $[0,2\pi]\times[0,2\pi]$, as we are. As discussed in [7], one may introduce a proportionality factor λ, and then require that $E = \lambda(t)G$, with λ satisfying a certain evolution equation. As in [7], for the sake of simplicity we study surfaces for which we may take $\lambda = 1$.*

From (2.2) and (2.3), we get

$$E_t = (\mathbf{X}_\alpha \cdot \mathbf{X}_\alpha)_t = 2\mathbf{X}_{\alpha t} \cdot \mathbf{X}_\alpha = 2\left((U\mathbf{n})_\alpha + (V_1\mathbf{X}_\alpha)_\alpha + (V_2\mathbf{X}_\beta)_\alpha\right) \cdot \mathbf{X}_\alpha, \tag{2.12}$$

$$G_t = (\mathbf{X}_\beta \cdot \mathbf{X}_\beta)_t = 2\mathbf{X}_{\beta t} \cdot \mathbf{X}_\beta = 2\Big((U\mathbf{n})_\beta + (V_1\mathbf{X}_\alpha)_\beta + (V_2\mathbf{X}_\beta)_\beta\Big) \cdot \mathbf{X}_\beta, \tag{2.13}$$

$$\begin{aligned} F_t &= (\mathbf{X}_\alpha \cdot \mathbf{X}_\beta)_t = \mathbf{X}_{\alpha t} \cdot \mathbf{X}_\beta + \mathbf{X}_{\beta t} \cdot \mathbf{X}_\alpha \\ &= \left((U\mathbf{n})_\alpha + (V_1\mathbf{X}_\alpha)_\alpha + (V_2\mathbf{X}_\beta)_\alpha\right) \cdot \mathbf{X}_\beta + \Big((U\mathbf{n})_\beta + (V_1\mathbf{X}_\alpha)_\beta + (V_2\mathbf{X}_\beta)_\beta\Big) \cdot \mathbf{X}_\alpha. \end{aligned} \tag{2.14}$$

Using $E = G$, $F = 0$ and the product rule to expand (2.12), (2.13), (2.14) and then plugging them into (2.11), we get

$$V_{1\alpha} - V_{2\beta} = \frac{U(L-N)}{E}, \tag{2.15}$$

$$V_{1\beta} + V_{2\alpha} = \frac{2UM}{E}. \tag{2.16}$$

Furthermore, these equations can be rewritten as

$$\Delta V_1 = \left(\frac{U(L-N)}{E}\right)_\alpha + \left(\frac{2UM}{E}\right)_\beta, \tag{2.17}$$

$$\Delta V_2 = \left(\frac{2UM}{E}\right)_\alpha - \left(\frac{U(L-N)}{E}\right)_\beta. \tag{2.18}$$

We select the unique solutions for V_1 and V_2 from (2.15) and (2.16) (or (2.17) and (2.18)) by setting their means to be zero. Define the inverse of Δ in terms of its Fourier symbol:

$$\widehat{(\Delta^{-1}f)}_{\mathbf{k}} = \begin{cases} 0, & |\mathbf{k}| = 0, \\ -\hat{f}_{\mathbf{k}}/|\mathbf{k}|^2, & |\mathbf{k}| \neq 0. \end{cases} \tag{2.19}$$

Then V_1 and V_2 are given by

$$V_1 = \Delta^{-1}\left(\left(\tfrac{U(L-N)}{E}\right)_\alpha + \left(\tfrac{2UM}{E}\right)_\beta\right), \tag{2.20}$$

$$V_2 = \Delta^{-1}\left(\left(\tfrac{2UM}{E}\right)_\alpha - \left(\tfrac{U(L-N)}{E}\right)_\beta\right). \tag{2.21}$$

We remark here that the formulas (2.20), (2.21) and the formulas in the preceding calculations are slightly different than those appearing in prior works such as [4] or [7]. This is because the choice not to normalize the tangent vectors, and using $\mathbf{X}_\alpha$ and $\mathbf{X}_\beta$ instead, in (2.3), simplifies some of the intermediate calculations.

We assume that the model problem represented by equations (2.3), (2.5), (2.6), (2.15) and (2.16) (or equivalently, (2.17) and (2.18)) is well-posed and has a sufficiently smooth solution $\mathbf{X}$ up to time $T > 0$. Furthermore, the interface $\mathbf{X}$ is assumed to be nonsingular under the $\boldsymbol{\alpha}$ parametrization. This implies that for all $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathbb{R}^2$, $t \in [0, T]$, and for some uniform constant $c > 0$ that does not depend on $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}'$ or t, we have

$$\left|\mathbf{X}(\boldsymbol{\alpha}, t) - \mathbf{X}(\boldsymbol{\alpha}', t)\right| \geq c\left|\boldsymbol{\alpha} - \boldsymbol{\alpha}'\right|. \tag{2.22}$$

The condition (2.22) is known as the chord-arc condition, and has been used in many analytical works, such as [24], [4]. Furthermore, we must assume that for the solution of the continuous problem, the component E of the first fundamental form remains bounded away from zero; thus, we assume that there exists a uniform constant c such that for all $\boldsymbol{\alpha} \in \mathbb{R}^2$ and for all t,

$$|E(\boldsymbol{\alpha}, t)| > c > 0. \tag{2.23}$$

3. SMALL-SCALE DECOMPOSITION

In this section, following [4], we rewrite the Birkhoff-Rott integral, given in (2.6), to separate the dominant terms at small scales, i.e., the higher order terms. We then rewrite the formula for the normal velocity U, separating out the highest-order terms, finding a formula that will be used in our numerical method. First we divide the kernel $-4\pi G(\mathbf{X} - \mathbf{X}') = \frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|^3}$ in (2.6) as follows:

$$\frac{\mathbf{X} - \mathbf{X}'}{|\mathbf{X} - \mathbf{X}'|^3} = \hat{\mathbf{H}} + \hat{\mathbf{G}} + \hat{\mathbf{K}}. \tag{3.1}$$

where

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2,\ \hat{\mathbf{G}} = \hat{\mathbf{G}}_{11} + \hat{\mathbf{G}}_{12} + \hat{\mathbf{G}}_{22}, \tag{3.2}$$

$$\hat{\mathbf{H}}_1 = \frac{\mathbf{X}'_\alpha(\alpha - \alpha')}{E'^{\frac{3}{2}}\,|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|^3},\ \hat{\mathbf{H}}_2 = \frac{\mathbf{X}'_\beta(\beta - \beta')}{E'^{\frac{3}{2}}\,|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|^3}, \tag{3.3}$$

$$\hat{\mathbf{G}}_{11} = \left(\frac{\mathbf{X}'_{\alpha\alpha}}{2E'^{\frac{3}{2}}} - \frac{3E'_\alpha\mathbf{X}'_\alpha}{4E'^{\frac{5}{2}}}\right)\frac{(\alpha - \alpha')^2}{|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|^3}, \tag{3.4}$$

$$\hat{\mathbf{G}}_{12} = \left(\frac{\mathbf{X}'_{\alpha\beta}}{E'^{\frac{3}{2}}} - \frac{3\left(E'_\alpha\mathbf{X}'_\beta + E'_\beta\mathbf{X}'_\alpha\right)}{4E'^{\frac{5}{2}}}\right)\frac{(\alpha - \alpha')(\beta - \beta')}{|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|^3}, \tag{3.5}$$

$$\hat{\mathbf{G}}_{22} = \left(\frac{\mathbf{X}'_{\beta\beta}}{2E'^{\frac{3}{2}}} - \frac{3E'_\beta\mathbf{X}'_\beta}{4E'^{\frac{5}{2}}}\right)\frac{(\beta - \beta')^2}{|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|^3} \tag{3.6}$$

$$\hat{\mathbf{K}} = \frac{\mathbf{X} - \mathbf{X}'}{|\mathbf{X} - \mathbf{X}'|^3} - \hat{\mathbf{H}} - \hat{\mathbf{G}}. \tag{3.7}$$

In this decomposition, $\hat{\mathbf{H}}$ is the most singular part of the kernel, $\hat{\mathbf{G}}$ is the weakly singular part and $\hat{\mathbf{K}}$ is the remaining nonsingular part.

We use $\mathbf{H}[\mathbf{X}]$, $\mathbf{G}[\mathbf{X}]$ and $\mathbf{K}[\mathbf{X}]$ to denote the following integral operators that act on a 3D vector-valued function $f(\boldsymbol{\alpha})$:

$$\mathbf{H}[\mathbf{X}]\boldsymbol{f}(\alpha, \beta) = \frac{1}{4\pi}PV\int \boldsymbol{f}(\alpha', \beta') \times \hat{\mathbf{H}}\, d\boldsymbol{\alpha}', \tag{3.8}$$

$$\mathbf{G}[\mathbf{X}]\boldsymbol{f}(\alpha, \beta) = \frac{1}{4\pi}PV\int \boldsymbol{f}(\alpha', \beta') \times \hat{\mathbf{G}}\, d\boldsymbol{\alpha}', \tag{3.9}$$

$$\mathbf{K}[\mathbf{X}]\boldsymbol{f}(\alpha, \beta) = \frac{1}{4\pi}PV\int \boldsymbol{f}(\alpha', \beta') \times \hat{\mathbf{K}}\, d\boldsymbol{\alpha}'. \tag{3.10}$$

Conditions for the finiteness of these integrals is discussed following (3.21). Then the Birkhoff-Riot integral (2.6) can be rewritten as

$$\mathbf{W} = -\mathbf{H}[\mathbf{X}]\boldsymbol{\eta} - \mathbf{G}[\mathbf{X}]\boldsymbol{\eta} - \mathbf{K}[\mathbf{X}]\boldsymbol{\eta}. \tag{3.11}$$

The classical Riesz transforms are defined as

$$\mathcal{H}_1 f(\alpha, \beta) = \frac{1}{2\pi}PV\int \frac{f(\alpha', \beta')(\alpha - \alpha')}{|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|^3}d\boldsymbol{\alpha}', \tag{3.12}$$

$$\mathcal{H}_2 f(\alpha, \beta) = \frac{1}{2\pi}PV\int \frac{f(\alpha', \beta')(\beta - \beta')}{|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|^3}d\boldsymbol{\alpha}', \tag{3.13}$$

and their Fourier symbols are, for $l \in \{1, 2\}$,

$$\left(\widehat{\mathcal{H}_l f}\right)_{\mathbf{k}} = \begin{cases} -i\frac{k_l}{|\mathbf{k}|}\hat{f}_{\mathbf{k}}, & |\mathbf{k}| \neq 0, \\ 0, & |\mathbf{k}| = 0. \end{cases} \tag{3.14}$$

Here the Fourier transform is defined by

$$\hat{f}_{\boldsymbol{k}} = \int e^{-i\boldsymbol{k}\cdot\boldsymbol{\alpha}} f(\boldsymbol{\alpha})d\boldsymbol{\alpha}. \tag{3.15}$$

Then, the leading order part of (2.6) corresponding to $\hat{\mathbf{H}}$ can be written as

$$\mathbf{H}[\mathbf{X}]\boldsymbol{\eta} = -\frac{1}{2}\left(\mathcal{H}_1\left(\frac{r_1\mathbf{n}}{\sqrt{E}}\right) + \mathcal{H}_2\left(\frac{r_2\mathbf{n}}{\sqrt{E}}\right)\right). \tag{3.16}$$

From the Fourier symbols, we see that $\mathcal{H}_l$, $l=1,2$ are bounded operators on the Soblev spaces $H^s(\mathbb{R}^2)$, for all $s\geq 0$.

For the weakly singular part, following [4], we define for a function f with zero mean (that is, $\int_0^{2\pi}\int_0^{2\pi} f(\boldsymbol{\alpha})d\boldsymbol{\alpha}=0$) the transforms $\mathcal{G}_{ij}$, $(i,j)=(1,1),(1,2),(2,2)$:

$$\mathcal{G}_{11}f(\alpha,\beta)=\frac{1}{4\pi}PV\int \frac{f(\alpha',\beta')(\alpha-\alpha')^2}{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3}d\boldsymbol{\alpha}', \tag{3.17}$$

$$\mathcal{G}_{12}f(\alpha,\beta)=\frac{1}{4\pi}PV\int \frac{f(\alpha',\beta')(\alpha-\alpha')(\beta-\beta')}{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3}d\boldsymbol{\alpha}', \tag{3.18}$$

$$\mathcal{G}_{22}f(\alpha,\beta)=\frac{1}{4\pi}PV\int \frac{f(\alpha',\beta')(\beta-\beta')^2}{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3}d\boldsymbol{\alpha}'. \tag{3.19}$$

The Fourier symbols of $\mathcal{G}_{ij}$, $(i,j)=(1,1),(1,2),(2,2)$ are:

$$\left(\widehat{\mathcal{G}_{11}f}\right)_{\mathbf{k}}=\frac{k_2^2}{2|\mathbf{k}|^3}\hat{f}_{\mathbf{k}},\ \left(\widehat{\mathcal{G}_{22}f}\right)_{\mathbf{k}}=\frac{k_1^2}{2|\mathbf{k}|^3}\hat{f}_{\mathbf{k}},\ \left(\widehat{\mathcal{G}_{12}f}\right)_{\mathbf{k}}=\left(\widehat{\mathcal{G}_{21}f}\right)_{\mathbf{k}}=-\frac{k_1k_2}{2|\mathbf{k}|^3}\hat{f}_{\mathbf{k}}, \tag{3.20}$$

for $\boldsymbol{k}\neq 0$. These follow from formulas [22] for the Fourier Transforms of so-called higher Riesz transforms, which gives the symbols for $\mathcal{G}_{12}$, $\mathcal{G}_{21}$, and $\mathcal{G}_{11}-\mathcal{G}_{22}$. The symbol for $\mathcal{G}_{11}+\mathcal{G}_{22}$ for a function of zero mean is determined from the well-known formula for the Fourier transform of $r^{-1}=|\boldsymbol{\alpha'}|^{-1}$. Combining these gives (3.20).

From the Fourier symbols, we see that the operators $\mathcal{G}_{ij}$ possess a smoothing property [2, 4] on the Soblev spaces $H^s(\mathbb{R}^2)$, $s\geq 0$, which will be explained more clearly in Section 7. The weakly singular part of (2.6) corresponding to $\hat{\mathbf{G}}$ can be written as

$$\begin{aligned}\mathbf{G}[\mathbf{X}]\boldsymbol{\eta}=&\mathcal{G}_{11}\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\alpha\alpha}}{2E^{\frac{3}{2}}}-\frac{3E_\alpha\mathbf{X}_\alpha}{4E^{\frac{5}{2}}}\right)\right)+\mathcal{G}_{22}\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\beta\beta}}{2E^{\frac{3}{2}}}-\frac{3E_\beta\mathbf{X}_\beta}{4E^{\frac{5}{2}}}\right)\right)\\&+\mathcal{G}_{12}\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\alpha\beta}}{E^{\frac{3}{2}}}-\frac{3\left(E_\alpha\mathbf{X}_\beta+E_\beta\mathbf{X}_\alpha\right)}{4E^{\frac{5}{2}}}\right)\right).\end{aligned} \tag{3.21}$$

Here it is understood that to make the integrals finite, the mean of the argument of $\mathcal{G}_{ij}$ is subtracted off.

We point out the following relationships between ∂_{α_l}, and $\mathcal{H}_l$ that will be useful to our reformulation of U:

Lemma 3.1. *The following equalities hold for the operators ∂_{α_l}, $\mathcal{H}_l$:*

$$\mathcal{H}_1\partial_{\alpha_2}=\partial_{\alpha_2}\mathcal{H}_1=\mathcal{H}_2\partial_{\alpha_1}=\partial_{\alpha_1}\mathcal{H}_2. \tag{3.22}$$

Proof. The results follow immediately from the Fourier symbols of ∂_{α_l} and $\mathcal{H}_l$. □

To facilitate later discussion, we introduce some notations. Define the operator

$$\Lambda=\mathcal{H}_1\partial_\alpha+\mathcal{H}_2\partial_\beta, \tag{3.23}$$

whose Fourier symbol is

$$\left(\widehat{\Lambda f}\right)_{\mathbf{k}}=|\mathbf{k}|\hat{f}_{\mathbf{k}}. \tag{3.24}$$

For an operator $\mathcal{A}$ acting on 3D vector-valued functions of the parameter $\boldsymbol{\alpha}$, and for such a function $\boldsymbol{f}(\boldsymbol{\alpha})$ and a scalar function $g(\boldsymbol{\alpha})$, we denote by $[\mathcal{A},\boldsymbol{f}]g$ the commutator $\mathcal{A}(\boldsymbol{f}g)-\boldsymbol{f}\mathcal{A}g$. For example, we have

$$[\mathcal{H}_i,\boldsymbol{f}]g=\mathcal{H}_i(\boldsymbol{f}g)-\boldsymbol{f}\mathcal{H}_ig,\quad i\in\{1,2\}, \tag{3.25}$$

$$[\mathcal{G}_{ij},\boldsymbol{f}]g=\mathcal{G}_{ij}(\boldsymbol{f}g)-\boldsymbol{f}\mathcal{G}_{ij}g,\quad (i,j)\in\{(1,1),(1,2),(2,2)\}. \tag{3.26}$$

Similar notations will be used to denote commutators corresponding to discretized operators in later discussions.

We take the dot product of (3.16) with $\mathbf{n}$, and we introduce commutators to write the result as

$$\mathbf{H}[\mathbf{X}]\boldsymbol{\eta}\cdot\mathbf{n} = -\frac{1}{2}\left(\mathcal{H}_1\left(\frac{r_1}{\sqrt{E}}\right)+\mathcal{H}_2\left(\frac{r_2}{\sqrt{E}}\right)\right)-\frac{1}{2}\mathbf{n}\cdot[\mathcal{H}_1,\mathbf{n}]\frac{r_1}{\sqrt{E}}-\frac{1}{2}\mathbf{n}\cdot[\mathcal{H}_2,\mathbf{n}]\frac{r_2}{\sqrt{E}}. \tag{3.27}$$

We also wish to rewrite (3.21). To do this, however, we need a few geometric identities, which follow from the isothermal parameterization:

$$(\mathbf{X}_\alpha\times\mathbf{X}_{\alpha\alpha})\cdot\mathbf{n} = -\frac{1}{2}E_\beta,\qquad (\mathbf{X}_\beta\times\mathbf{X}_{\alpha\alpha})\cdot\mathbf{n} = -\frac{1}{2}E_\alpha,$$
$$(\mathbf{X}_\alpha\times\mathbf{X}_{\alpha\beta})\cdot\mathbf{n} = \frac{1}{2}E_\alpha,\qquad (\mathbf{X}_\beta\times\mathbf{X}_{\alpha\beta})\cdot\mathbf{n} = -\frac{1}{2}E_\beta,$$
$$(\mathbf{X}_\alpha\times\mathbf{X}_{\beta\beta})\cdot\mathbf{n} = \frac{1}{2}E_\beta,\qquad (\mathbf{X}_\beta\times\mathbf{X}_{\beta\beta})\cdot\mathbf{n} = \frac{1}{2}E_\alpha.$$

Using these equations and introducing commutators as is convenient, we take the dot product of (3.21) with $\mathbf{n}$ and write the result as follows:

(3.28)

$$\begin{aligned}
\mathbf{G}[\mathbf{X}]\boldsymbol{\eta}\cdot\mathbf{n} =& \mathcal{G}_{11}\left(\frac{\frac{1}{2}r_1E_\alpha+\frac{1}{4}r_2E_\beta}{E^{\frac{3}{2}}}\right)+\mathcal{G}_{22}\left(\frac{\frac{1}{4}r_1E_\alpha+\frac{1}{2}r_2E_\beta}{E^{\frac{3}{2}}}\right)+\mathcal{G}_{12}\left(\frac{\frac{1}{4}r_1E_\beta+\frac{1}{4}r_2E_\alpha}{E^{\frac{3}{2}}}\right)\\
&-[\mathcal{G}_{11},\mathbf{n}]\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\alpha\alpha}}{2E^{\frac{3}{2}}}-\frac{3E_\alpha\mathbf{X}_\alpha}{4E^{\frac{5}{2}}}\right)\right)-[\mathcal{G}_{22},\mathbf{n}]\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\beta\beta}}{2E^{\frac{3}{2}}}-\frac{3E_\beta\mathbf{X}_\beta}{4E^{\frac{5}{2}}}\right)\right)\\
&-[\mathcal{G}_{12},\mathbf{n}]\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\alpha\beta}}{E^{\frac{3}{2}}}-\frac{3\left(E_\alpha\mathbf{X}_\beta+E_\beta\mathbf{X}_\alpha\right)}{4E^{\frac{5}{2}}}\right)\right)\\
=&\frac{E_\alpha}{2E}\mathcal{G}_{11}(\frac{r_1}{\sqrt{E}})+\frac{E_\beta}{4E}\mathcal{G}_{11}(\frac{r_2}{\sqrt{E}})+\frac{E_\alpha}{4E}\mathcal{G}_{22}(\frac{r_1}{\sqrt{E}})+\frac{E_\beta}{2E}\mathcal{G}_{22}(\frac{r_2}{\sqrt{E}})+\frac{E_\beta}{4E}\mathcal{G}_{12}(\frac{r_1}{\sqrt{E}})+\frac{E_\alpha}{4E}\mathcal{G}_{12}(\frac{r_2}{\sqrt{E}})\\
&-[\mathcal{G}_{11},\mathbf{n}]\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\alpha\alpha}}{2E^{\frac{3}{2}}}-\frac{3E_\alpha\mathbf{X}_\alpha}{4E^{\frac{5}{2}}}\right)\right)-[\mathcal{G}_{22},\mathbf{n}]\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\beta\beta}}{2E^{\frac{3}{2}}}-\frac{3E_\beta\mathbf{X}_\beta}{4E^{\frac{5}{2}}}\right)\right)\\
&-[\mathcal{G}_{12},\mathbf{n}]\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\alpha\beta}}{E^{\frac{3}{2}}}-\frac{3\left(E_\alpha\mathbf{X}_\beta+E_\beta\mathbf{X}_\alpha\right)}{4E^{\frac{5}{2}}}\right)\right)+[\mathcal{G}_{11},\frac{E_\alpha}{2E}]\frac{r_1}{\sqrt{E}}+[\mathcal{G}_{11},\frac{E_\beta}{4E}]\frac{r_2}{\sqrt{E}}\\
&+[\mathcal{G}_{22},\frac{E_\alpha}{4E}]\frac{r_1}{\sqrt{E}}+[\mathcal{G}_{22},\frac{E_\beta}{2E}]\frac{r_2}{\sqrt{E}}+[\mathcal{G}_{12},\frac{E_\beta}{4E}]\frac{r_1}{\sqrt{E}}+[\mathcal{G}_{12},\frac{E_\alpha}{4E}]\frac{r_2}{\sqrt{E}}.
\end{aligned}$$

Using the definition $U=\mathbf{W}\cdot\mathbf{n}$ and combining (3.11), (3.27) and (3.28), we have the following formula for U:

(3.29)

$$\begin{aligned}
U =& \frac{1}{2}\left(\mathcal{H}_1\left(\frac{r_1}{\sqrt{E}}\right)+\mathcal{H}_2\left(\frac{r_2}{\sqrt{E}}\right)\right)+\frac{1}{2}\mathbf{n}\cdot[\mathcal{H}_1,\mathbf{n}]\frac{r_1}{\sqrt{E}}+\frac{1}{2}\mathbf{n}\cdot[\mathcal{H}_2,\mathbf{n}]\frac{r_2}{\sqrt{E}}\\
&-\frac{E_\alpha}{2E}\mathcal{G}_{11}(\frac{r_1}{\sqrt{E}})-\frac{E_\beta}{4E}\mathcal{G}_{11}(\frac{r_2}{\sqrt{E}})-\frac{E_\alpha}{4E}\mathcal{G}_{22}(\frac{r_1}{\sqrt{E}})-\frac{E_\beta}{2E}\mathcal{G}_{22}(\frac{r_2}{\sqrt{E}})-\frac{E_\beta}{4E}\mathcal{G}_{12}(\frac{r_1}{\sqrt{E}})-\frac{E_\alpha}{4E}\mathcal{G}_{12}(\frac{r_2}{\sqrt{E}})\\
&+[\mathcal{G}_{11},\mathbf{n}]\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\alpha\alpha}}{2E^{\frac{3}{2}}}-\frac{3E_\alpha\mathbf{X}_\alpha}{4E^{\frac{5}{2}}}\right)\right)+[\mathcal{G}_{22},\mathbf{n}]\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\beta\beta}}{2E^{\frac{3}{2}}}-\frac{3E_\beta\mathbf{X}_\beta}{4E^{\frac{5}{2}}}\right)\right)\\
&+[\mathcal{G}_{12},\mathbf{n}]\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\alpha\beta}}{E^{\frac{3}{2}}}-\frac{3\left(E_\alpha\mathbf{X}_\beta+E_\beta\mathbf{X}_\alpha\right)}{4E^{\frac{5}{2}}}\right)\right)-[\mathcal{G}_{11},\frac{E_\alpha}{2E}]\frac{r_1}{\sqrt{E}}-[\mathcal{G}_{11},\frac{E_\beta}{4E}]\frac{r_2}{\sqrt{E}}\\
&-[\mathcal{G}_{22},\frac{E_\alpha}{4E}]\frac{r_1}{\sqrt{E}}-[\mathcal{G}_{22},\frac{E_\beta}{2E}]\frac{r_2}{\sqrt{E}}-[\mathcal{G}_{12},\frac{E_\beta}{4E}]\frac{r_1}{\sqrt{E}}-[\mathcal{G}_{12},\frac{E_\alpha}{4E}]\frac{r_2}{\sqrt{E}}-\mathbf{K}[\mathbf{X}]\boldsymbol{\eta}\cdot\mathbf{n}.
\end{aligned}$$

We use S to denote the part of (3.29) which does not include the commutators and which does not include $\mathbf{K}$; that is,

$$
\begin{aligned}
S =& \frac{1}{2}\left(\mathcal{H}_1\left(\frac{r_1}{\sqrt{E}}\right)+\mathcal{H}_2\left(\frac{r_2}{\sqrt{E}}\right)\right)-\frac{E_\alpha}{2E}\mathcal{G}_{11}(\frac{r_1}{\sqrt{E}})-\frac{E_\beta}{4E}\mathcal{G}_{11}(\frac{r_2}{\sqrt{E}}) \\
&-\frac{E_\alpha}{4E}\mathcal{G}_{22}(\frac{r_1}{\sqrt{E}})-\frac{E_\beta}{2E}\mathcal{G}_{22}(\frac{r_2}{\sqrt{E}})-\frac{E_\beta}{4E}\mathcal{G}_{12}(\frac{r_1}{\sqrt{E}})-\frac{E_\alpha}{4E}\mathcal{G}_{12}(\frac{r_2}{\sqrt{E}}).
\end{aligned}
\tag{3.30}
$$

We will further decompose S later (to be precise, in the next section, we will give a decomposition of ΔS), but for now, we will mention that it includes the leading-order part of U.

Use T to denote the sum of the commutator terms in (3.29):

(3.31)

$$
\begin{aligned}
T =& \frac{1}{2}\mathbf{n}\cdot[\mathcal{H}_1,\mathbf{n}]\frac{r_1}{\sqrt{E}}+\frac{1}{2}\mathbf{n}\cdot[\mathcal{H}_2,\mathbf{n}]\frac{r_2}{\sqrt{E}}+[\mathcal{G}_{11},\mathbf{n}]\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\alpha\alpha}}{2E^{\frac{3}{2}}}-\frac{3E_\alpha\mathbf{X}_\alpha}{4E^{\frac{5}{2}}}\right)\right)+[\mathcal{G}_{22},\mathbf{n}]\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\beta\beta}}{2E^{\frac{3}{2}}}-\frac{3E_\beta\mathbf{X}_\beta}{4E^{\frac{5}{2}}}\right)\right) \\
&+[\mathcal{G}_{12},\mathbf{n}]\left(\boldsymbol{\eta}\times\left(\frac{\mathbf{X}_{\alpha\beta}}{E^{\frac{3}{2}}}-\frac{3\left(E_\alpha\mathbf{X}_\beta+E_\beta\mathbf{X}_\alpha\right)}{4E^{\frac{5}{2}}}\right)\right)-[\mathcal{G}_{11},\frac{E_\alpha}{2E}]\frac{r_1}{\sqrt{E}}-[\mathcal{G}_{11},\frac{E_\beta}{4E}]\frac{r_2}{\sqrt{E}}-[\mathcal{G}_{22},\frac{E_\alpha}{4E}]\frac{r_1}{\sqrt{E}} \\
&-[\mathcal{G}_{22},\frac{E_\beta}{2E}]\frac{r_2}{\sqrt{E}}-[\mathcal{G}_{12},\frac{E_\beta}{4E}]\frac{r_1}{\sqrt{E}}-[\mathcal{G}_{12},\frac{E_\alpha}{4E}]\frac{r_2}{\sqrt{E}}.
\end{aligned}
$$

We also use the notation K to denote the last remainder:

$$
K=-\mathbf{K}[\mathbf{X}]\boldsymbol{\eta}\cdot\mathbf{n}. \tag{3.32}
$$

With (3.30), (3.31) and (3.32), the formula for U can be rewritten as

$$
U=S+T+K. \tag{3.33}
$$

In our numerical method we need to add a corrector term to balance off the error generated by $\hat{\mathbf{K}}$. As suggested in [10, 11, 20], we make use of the following identity:

$$
\int\nabla_{\mathbf{x}}G(\mathbf{X}-\mathbf{X}')\times\mathbf{N}'d\boldsymbol{\alpha}'=0. \tag{3.34}
$$

Define

$$
C_{\mathbf{n}}=\frac{\boldsymbol{\eta}(\boldsymbol{\alpha})}{|\mathbf{N}(\boldsymbol{\alpha})|}\cdot\int\nabla_{\mathbf{x}}G(\mathbf{X}(\boldsymbol{\alpha})-\mathbf{X}(\boldsymbol{\alpha}'))\times\mathbf{N}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}'. \tag{3.35}
$$

From (3.34) we immediately see that $C_{\mathbf{n}}=0$. We do a similar small-scale decomposition of $C_{\mathbf{n}}$ based on (3.1):

(3.36)

$$
\begin{aligned}
C_{\mathbf{n}} =& \frac{\boldsymbol{\eta}(\boldsymbol{\alpha})}{|\mathbf{N}(\boldsymbol{\alpha})|}\cdot\int\nabla_{\mathbf{x}}G(\mathbf{X}(\boldsymbol{\alpha})-\mathbf{X}(\boldsymbol{\alpha}'))\times\mathbf{N}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}' \\
=& -\frac{\boldsymbol{\eta}(\boldsymbol{\alpha})}{4\pi|\mathbf{N}(\boldsymbol{\alpha})|}\cdot\int\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|^3}\times\mathbf{N}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}' \\
=& -\frac{\boldsymbol{\eta}(\boldsymbol{\alpha})}{4\pi|\mathbf{N}(\boldsymbol{\alpha})|}\cdot\int\left(\hat{\mathbf{H}}+\hat{\mathbf{G}}+\hat{\mathbf{K}}\right)\times\mathbf{N}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}' \\
=& -\frac{\boldsymbol{\eta}}{2|\mathbf{N}|}\cdot\left(\mathcal{H}_1\left(\frac{\mathbf{X}_\alpha}{E^{\frac{3}{2}}}\times\mathbf{N}\right)+\mathcal{H}_2\left(\frac{\mathbf{X}_\beta}{E^{\frac{3}{2}}}\times\mathbf{N}\right)\right)-\frac{\boldsymbol{\eta}}{|\mathbf{N}|}\cdot\mathcal{G}_{11}\left(\left(\frac{\mathbf{X}_{\alpha\alpha}}{2E^{\frac{3}{2}}}-\frac{3E_\alpha\mathbf{X}_\alpha}{4E^{\frac{5}{2}}}\right)\times\mathbf{N}\right) \\
&-\frac{\boldsymbol{\eta}}{|\mathbf{N}|}\cdot\mathcal{G}_{12}\left(\left(\frac{\mathbf{X}_{\alpha\beta}}{E^{\frac{3}{2}}}-\frac{3\left(E_\alpha\mathbf{X}_\beta+E_\beta\mathbf{X}_\alpha\right)}{4E^{\frac{5}{2}}}\right)\times\mathbf{N}\right)-\frac{\boldsymbol{\eta}}{|\mathbf{N}|}\cdot\mathcal{G}_{22}\left(\left(\frac{\mathbf{X}_{\beta\beta}}{2E^{\frac{3}{2}}}-\frac{3E_\beta\mathbf{X}_\beta}{4E^{\frac{5}{2}}}\right)\times\mathbf{N}\right) \\
&-\frac{\boldsymbol{\eta}}{|\mathbf{N}|}\cdot\mathbf{K}[\mathbf{X}]\mathbf{N}.
\end{aligned}
$$

Since $C_{\mathbf{n}} = 0$, we are able to subtract it from U in (3.33) without changing the value of U; we thus obtain the desired decomposition of U:

$$U = S + T + K - C_{\mathbf{n}}. \tag{3.37}$$

4. Evolution equations

For purposes that will be made clear in our numerical scheme, we consider $\mathbf{X}$, $\mathbf{X}_\alpha$, $\mathbf{X}_\beta$ and κ as independent variables. Recall that the evolution equation for $\mathbf{X}$ is given by

$$\frac{d\mathbf{X}}{dt} = \mathbf{V} = U\mathbf{n} + V_1\mathbf{X}_\alpha + V_2\mathbf{X}_\beta, \tag{4.1}$$

where U can be calculated from (3.37), (3.30), (3.31), (3.32), (3.36) and V_1, V_2 are determined by (2.20) and (2.21). We immediately obtain the evolution equations for $\mathbf{X}_\alpha$ and $\mathbf{X}_\beta$:

$$\mathbf{X}_{\alpha t} = \mathbf{V}_\alpha = \left(U\mathbf{n} + V_1\mathbf{X}_\alpha + V_2\mathbf{X}_\beta\right)_\alpha, \tag{4.2}$$

$$\mathbf{X}_{\beta t} = \mathbf{V}_\beta = \left(U\mathbf{n} + V_1\mathbf{X}_\alpha + V_2\mathbf{X}_\beta\right)_\beta. \tag{4.3}$$

Now we consider the evolution equation for κ. As shown in [4], from (2.9) and (4.1) we get

$$(\sqrt{E}\kappa)_t = \frac{1}{2\sqrt{E}}\left(U_\alpha + V_1 L + V_2 M\right)_\alpha + \frac{1}{2\sqrt{E}}\left(U_\beta + V_1 M + V_2 N\right)_\beta, \tag{4.4}$$

$$E_t = 2V_{1\alpha}E + V_1 E_\alpha + V_2 E_\beta - 2UL. \tag{4.5}$$

Upon combining these, we find

$$\kappa_t = \frac{1}{2E}\Delta U + \frac{\kappa}{2E}\left(2UL - 2V_{1\alpha}E - V_1 E_\alpha - V_2 E_\beta\right) + \frac{1}{2E}\Big(\left(V_1 L + V_2 M\right)_\alpha + \left(V_1 M + V_2 N\right)_\beta\Big). \tag{4.6}$$

From (3.37), we get

$$\Delta U = \Delta S + \Delta T + \Delta K - \Delta C_n. \tag{4.7}$$

Then, from (3.30), and since $\Delta = -\Lambda^2$, $\Lambda\mathcal{H}_1 = -\partial_\alpha$, and $\Lambda\mathcal{H}_2 = -\partial_\beta$, we get

$$\begin{aligned}\Delta S = {} & \frac{1}{2}\Lambda\left(\partial_\alpha\left(\frac{r_1}{\sqrt{E}}\right) + \partial_\beta\left(\frac{r_2}{\sqrt{E}}\right)\right) \\ & - \Delta\left(\frac{E_\alpha}{2E}\mathcal{G}_{11} + \frac{E_\alpha}{4E}\mathcal{G}_{22} + \frac{E_\beta}{4E}\mathcal{G}_{12}\right)\left(\frac{r_1}{\sqrt{E}}\right) - \Delta\left(\frac{E_\beta}{4E}\mathcal{G}_{11} + \frac{E_\beta}{2E}\mathcal{G}_{22} + \frac{E_\alpha}{4E}\mathcal{G}_{12}\right)\left(\frac{r_2}{\sqrt{E}}\right).\end{aligned} \tag{4.8}$$

Next, we substitute (2.7) into the first terms on the right-hand side of (4.8):

$$\begin{aligned}\Delta S = {} & \frac{B}{2}\Lambda\left(\left(\frac{\kappa_\alpha}{\sqrt{E}}\right)_\alpha + \left(\frac{\kappa_\beta}{\sqrt{E}}\right)_\beta\right) - \Delta\left(\frac{E_\alpha}{2E}\mathcal{G}_{11} + \frac{E_\alpha}{4E}\mathcal{G}_{22} + \frac{E_\beta}{4E}\mathcal{G}_{12}\right)\left(\frac{r_1}{\sqrt{E}}\right) \\ & - \Delta\left(\frac{E_\beta}{4E}\mathcal{G}_{11} + \frac{E_\beta}{2E}\mathcal{G}_{22} + \frac{E_\alpha}{4E}\mathcal{G}_{12}\right)\left(\frac{r_2}{\sqrt{E}}\right) = -\frac{B}{2\sqrt{E}}\Lambda^3\kappa + R,\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}R = {} & \frac{B}{2}\left[\Lambda\partial_\alpha, \frac{1}{\sqrt{E}}\right]\kappa_\alpha + \frac{B}{2}\left[\Lambda\partial_\beta, \frac{1}{\sqrt{E}}\right]\kappa_\beta \\ & - \Delta\left(\frac{E_\alpha}{2E}\mathcal{G}_{11} + \frac{E_\alpha}{4E}\mathcal{G}_{22} + \frac{E_\beta}{4E}\mathcal{G}_{12}\right)\left(\frac{r_1}{\sqrt{E}}\right) - \Delta\left(\frac{E_\beta}{4E}\mathcal{G}_{11} + \frac{E_\beta}{2E}\mathcal{G}_{22} + \frac{E_\alpha}{4E}\mathcal{G}_{12}\right)\left(\frac{r_2}{\sqrt{E}}\right).\end{aligned} \tag{4.10}$$

As a result, ΔU can be written as

$$\Delta U = -\frac{B}{2\sqrt{E}}\Lambda^3\kappa + R + \Delta T + \Delta K - \Delta C_n. \tag{4.11}$$

Plugging (4.11) into (4.6), we now rewrite the evolution equation for κ:

$$\begin{aligned}\kappa_t =& -\frac{B}{4E^{3/2}}\Lambda^3\kappa + \frac{1}{2E}\left(R + \Delta T + \Delta K - \Delta C_n\right) \\ &+ \frac{\kappa}{2E}\left(2UL - 2V_{1\alpha}E - V_1E_\alpha - V_2E_\beta\right) + \frac{1}{2E}\Big((V_1L + V_2M)_\alpha + (V_1M + V_2N)_\beta\Big).\end{aligned} \tag{4.12}$$

5. Numerical method

In this section we describe our numerical method. A semi-discretization scheme is used here to facilitate the stability analysis, i.e., the time axis is not discretized. To discretize the spatial differential operators, we lay a uniform Cartesian grid with grid width $h = \frac{2\pi}{N}$ on the $\boldsymbol{\alpha}$ domain. As mentioned earlier, we consider $\mathbf{X}$, $\mathbf{X}_\alpha$, $\mathbf{X}_\beta$ and κ as independent variables and the corresponding numerical approximations $\mathbf{X}_h$, $\mathbf{X}_{\alpha h}$, $\mathbf{X}_{\beta h}$ and κ_h are solved from their evolution equations obtained by discretizing (4.1), (4.2), (4.3) and (4.12). We use U_h, V_{1h} and V_{2h} to denote the numerical approximations of U, V_1 and V_2. Other notations are defined similarly.

5.1. **Discretized operators.** The spatial derivatives are discretized spectrally. Recall that for a 2π-periodic function, $u(\boldsymbol{\alpha})$, its discrete Fourier transform is given by

$$\tilde{u}_{\mathbf{k}} = h^2 \sum_{(j_1,j_2)=(-\frac{N}{2}+1,-\frac{N}{2}+1)}^{(\frac{N}{2},\frac{N}{2})} u(\boldsymbol{\alpha}_{\mathbf{j}})e^{-i\mathbf{k}\cdot\mathbf{j}h}, \tag{5.1}$$

where $\mathbf{j} = (j_1, j_2)$, $k_1, k_2 = -\frac{N}{2}+1, \cdots, \frac{N}{2}$. The inverse formula is

$$u_{\mathbf{j}} = \frac{1}{(2\pi)^2} \sum_{(k_1,k_2)=(-\frac{N}{2}+1,-\frac{N}{2}+1)}^{(\frac{N}{2},\frac{N}{2})} \tilde{u}(\boldsymbol{\alpha}_{\mathbf{j}})e^{i\mathbf{k}\cdot\mathbf{j}h}. \tag{5.2}$$

Using the discrete Fourier transform, we define the discrete derivatives D_{1h}, D_{2h} by

$$(\widetilde{D_{lh}f_h})_{\mathbf{k}} = ik_l\tilde{f}_{h\mathbf{k}}. \tag{5.3}$$

Because of the isothermal assumption, of the unit vectors and the fundamental forms, only $\mathbf{n}$, E, L, M and N appear in the evolution equations (4.1), (4.2), (4.3) and (4.12). They are calculated from the independent variables $\mathbf{X}_h$, $\mathbf{X}_{\alpha h}$, $\mathbf{X}_{\beta h}$, and κ_h as follows:

$$\begin{aligned}&\mathbf{n}_h = \frac{\mathbf{X}_{\alpha h}\times\mathbf{X}_{\beta h}}{E_h}, \quad E_h = \mathbf{X}_{\alpha h}\cdot\mathbf{X}_{\alpha h}, \\ &L_h = -\mathbf{X}_{\alpha h}\cdot D_{1h}\mathbf{n}_h, \quad M_h = -\mathbf{X}_{\alpha h}\cdot D_{2h}\mathbf{n}_h, \quad N_h = -\mathbf{X}_{\beta h}\cdot D_{2h}\mathbf{n}_h.\end{aligned} \tag{5.4}$$

The discrete dipole strength $\boldsymbol{\eta}_h$ is calculated from

$$\boldsymbol{\eta}_h = r_{1h}\mathbf{X}_{\beta h} - r_{2h}\mathbf{X}_{\alpha h}, \tag{5.5}$$

where

$$r_{1h} = BD_{1h}\kappa_h, \qquad r_{2h} = BD_{2h}\kappa_h. \tag{5.6}$$

We need to derive the evolution equations for $\mathbf{X}_h$, $\mathbf{X}_{\alpha h}$, $\mathbf{X}_{\beta h}$ and κ_h. First consider $\mathbf{X}_h$. Recalling (4.1), we introduce $\mathbf{V}_h$,

$$\mathbf{V}_h = U_h\mathbf{n}_h + V_{1h}\mathbf{X}_{\alpha h} + V_{2h}\mathbf{X}_{\beta h} \tag{5.7}$$

In the above equation, we need to discretize (3.37) to calculate U_h, which requires discretizing (3.30), (3.31), (3.32) and (3.36). In (3.30), we discretize the Riesz transforms $\mathcal{H}_1$ and $\mathcal{H}_2$ spectrally without filtering. The discretized operators $\mathcal{H}^s_{lh}$, $l=1,2$ can be expressed in terms of their discrete Fourier symbols:

$$\left(\widetilde{\mathcal{H}^s_{lh} f_h}\right)_{\mathbf{k}} = -i\frac{k_l}{|\mathbf{k}|}\tilde{f}_{h\mathbf{k}}, \quad l=1,2, \quad k_1,k_2=-\frac{N}{2}+1,\cdots,\frac{N}{2}\neq 0, \tag{5.8}$$

and zero for $\boldsymbol{k}=0$.

The $\mathcal{G}_{ij}$ operators are also discretized similarly and the corresponding discrete operators $\mathcal{G}^s_{ij,h}$ can be expressed as follows:

$$\left(\widetilde{\mathcal{G}^s_{11,h} f_h}\right)_{\mathbf{k}} = \frac{k_2^2}{2\,|\mathbf{k}|^3}\hat{f}_{h\mathbf{k}},\ \left(\widetilde{\mathcal{G}^s_{22,h} f_h}\right)_{\mathbf{k}} = \frac{k_1^2}{2\,|\mathbf{k}|^3}\hat{f}_{h\mathbf{k}},\ \left(\widetilde{\mathcal{G}^s_{12,h} f_h}\right)_{\mathbf{k}} = \left(\widetilde{\mathcal{G}^s_{21,h} f_h}\right)_{\mathbf{k}} = -\frac{k_1k_2}{2\,|\mathbf{k}|^3}\hat{f}_{h\mathbf{k}}, \tag{5.9}$$

for $k_1,k_2=-\frac{N}{2}+1,\cdots,\frac{N}{2}\neq 0$.

Naturally, from (5.3) and (5.8), the spectral discretization of the operator Λ defined in (3.23) is

$$\Lambda_h = \mathcal{H}^s_{1h}D_{1h} + \mathcal{H}^s_{2h}D_{2h}, \tag{5.10}$$

and its discrete Fourier symbol is

$$\left(\widetilde{\Lambda_h f_h}\right)_{\mathbf{k}} = |\mathbf{k}|\tilde{f}_{h\mathbf{k}}, \quad l=1,2. \tag{5.11}$$

The discrete approximation of the higher order term S is calculated as follows:

$$\begin{aligned} S_h =& \frac{1}{2}\left(\mathcal{H}^s_{1h}\left(\frac{r_{1h}}{\sqrt{E_h}}\right) + \mathcal{H}^s_{2h}\left(\frac{r_{2h}}{\sqrt{E_h}}\right)\right) - \frac{D_{1h}E_h}{2E_h}\mathcal{G}^s_{11,h}(\frac{r_{1h}}{\sqrt{E_h}}) - \frac{D_{2h}E_h}{4E_h}\mathcal{G}^s_{11,h}(\frac{r_{2h}}{\sqrt{E_h}}) \\ & - \frac{D_{1h}E_h}{4E_h}\mathcal{G}^s_{22,h}(\frac{r_{1h}}{\sqrt{E_h}}) - \frac{D_{2h}E_h}{2E_h}\mathcal{G}^s_{22,h}(\frac{r_{2h}}{\sqrt{E_h}}) - \frac{D_{2h}E_h}{4E_h}\mathcal{G}^s_{12,h}(\frac{r_{1h}}{\sqrt{E_h}}) - \frac{D_{1h}E_h}{4E_h}\mathcal{G}^s_{12,h}(\frac{r_{2h}}{\sqrt{E_h}}). \end{aligned} \tag{5.12}$$

The integrals in T and K are discretized using the standard point vortex method. For a function $f_h(\boldsymbol{\alpha})$ defined on $\mathbb{R}^2_h$, the standard point vortex approximation of the principle integral of f_h on $\mathbb{R}^2_h$ is defined as

$$P.V.\int_h^p f_h(\mathbf{x_i}) = P.V.\sum_{\mathbf{j}\neq\mathbf{i},\ \mathbf{j}\in\mathbb{Z}^2} f_h(\mathbf{X_j})h^2. \tag{5.13}$$

We often use the notation $\int_h^p$ with superscript p to indicate a standard point vortex integral. Also, we use $\mathcal{H}^p_{lh}$ and $\mathcal{G}^p_{ij,h}$ to denote the standard point vortex approximations of $\mathcal{H}_l$ and $\mathcal{G}_{ij}$.

The approximation of T is obtained from discretizing (3.31):

$$\begin{aligned} T_h =& \frac{1}{2}\mathbf{n}_h\cdot[\mathcal{H}^p_{1h},\mathbf{n}_h]\frac{r_{1h}}{\sqrt{E_h}} + \frac{1}{2}\mathbf{n}_h\cdot[\mathcal{H}^p_{2h},\mathbf{n}_h]\frac{r_{2h}}{\sqrt{E_h}} \\ & + [\mathcal{G}^p_{11,h},\mathbf{n}_h]\left(\eta_h\times\left(\frac{D_{1h}\mathbf{X}_{\alpha h}}{2E_h^{\frac{3}{2}}} - \frac{3\mathbf{X}_{\alpha h}D_{1h}E_h}{4E_h^{\frac{5}{2}}}\right)\right) + [\mathcal{G}^p_{22,h},\mathbf{n}_h]\left(\eta_h\times\left(\frac{D_{2h}\mathbf{X}_{\beta h}}{2E_h^{\frac{3}{2}}} - \frac{3\mathbf{X}_{\beta h}D_{2h}E_h}{4E_h^{\frac{5}{2}}}\right)\right) \\ & + [\mathcal{G}^p_{12,h},\mathbf{n}_h]\left(\eta_h\times\left(\frac{D_{1h}\mathbf{X}_{\beta h}}{E_h^{\frac{3}{2}}} - \frac{3\left(\mathbf{X}_{\beta h}D_{1h}E_h + \mathbf{X}_{\alpha h}D_{2h}E_h\right)}{4E_h^{\frac{5}{2}}}\right)\right) \\ & - [\mathcal{G}^p_{11,h},\frac{D_{1h}E_h}{2E_h}]\frac{r_{1h}}{\sqrt{E_h}} - [\mathcal{G}^p_{11,h},\frac{D_{2h}E_h}{4E_h}]\frac{r_{2h}}{\sqrt{E_h}} - [\mathcal{G}^p_{22,h},\frac{D_{1h}E_h}{4E_h}]\frac{r_{1h}}{\sqrt{E_h}} \\ & - [\mathcal{G}^p_{22,h},\frac{D_{2h}E_h}{2E_h}]\frac{r_{2h}}{\sqrt{E_h}} - [\mathcal{G}^p_{12,h},\frac{D_{2h}E_h}{4E_h}]\frac{r_{1h}}{\sqrt{E_h}} - [\mathcal{G}^p_{12,h},\frac{D_{1h}E_h}{4E_h}]\frac{r_{2h}}{\sqrt{E_h}}. \end{aligned} \tag{5.14}$$

For the operators appearing inside commutators (such as Riesz transforms), note that we have specified point-vortex versions of the discretized operators rather than spectral versions. This is so that we may use the smoothing properties of these commutators, which are detailed below in Section 7. Another possible choice here would be to use spectral versions, but to introduce a dealiasing filter, as described in [14]; we choose instead to work with a version of the method which requires no filtering.

Discretizing (3.32), we get

$$K_h = -\mathbf{K}_h^p[\mathbf{X}_h]\boldsymbol{\eta}_h = -\frac{1}{4\pi} P.V. \sum_{\mathbf{j}\neq\mathbf{i},\, \mathbf{j}\in\mathbb{Z}^2} \boldsymbol{\eta}_h(\mathbf{X}_{\mathbf{j}h}) \times \hat{\mathbf{K}}_h h^2, \tag{5.15}$$

where $\hat{\mathbf{K}}_h$ is defined as follows:

$$\hat{\mathbf{K}}_h = \frac{\mathbf{X}_h - \mathbf{X}'_h}{\left|\mathbf{X}_h - \mathbf{X}'_h\right|^3} - \hat{\mathbf{H}}_h - \hat{\mathbf{G}}_h, \tag{5.16}$$

$$\hat{\mathbf{H}}_h = \hat{\mathbf{H}}_{1h} + \hat{\mathbf{H}}_{2h}, \tag{5.17}$$

$$\hat{\mathbf{H}}_{1h} = \frac{\mathbf{X}'_{\alpha h}(\alpha-\alpha')}{E_h'^{\frac{3}{2}}\,|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3},\ \hat{\mathbf{H}}_{2h} = \frac{\mathbf{X}'_{\beta h}(\beta-\beta')}{E_h'^{\frac{3}{2}}\,|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3}, \tag{5.18}$$

$$\hat{\mathbf{G}}_h = \hat{\mathbf{G}}_{11,h} + \hat{\mathbf{G}}_{12,h} + \hat{\mathbf{G}}_{22,h}, \tag{5.19}$$

$$\hat{\mathbf{G}}_{11,h} = \left(\frac{D_{1h}\mathbf{X}'_{\alpha h}}{2E_h'^{\frac{3}{2}}} - \frac{3\mathbf{X}'_{\alpha h} D_{1h} E'_h}{4E_h'^{\frac{5}{2}}} \right) \frac{(\alpha-\alpha')^2}{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3}, \tag{5.20}$$

$$\hat{\mathbf{G}}_{12,h} = \left(\frac{D_{1h}\mathbf{X}'_{\beta h}}{2E_h'^{\frac{3}{2}}} - \frac{3\left(\mathbf{X}'_{\beta h} D_{1h} E'_h + \mathbf{X}'_{\alpha h} D_{2h} E'_h\right)}{4E_h'^{\frac{5}{2}}} \right) \frac{(\alpha-\alpha')(\beta-\beta')}{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3}, \tag{5.21}$$

$$\hat{\mathbf{G}}_{22,h} = \left(\frac{D_{2h}\mathbf{X}'_{\beta h}}{2E_h'^{\frac{3}{2}}} - \frac{3\mathbf{X}'_{\beta h} D_{2h} E'_h}{4E_h'^{\frac{5}{2}}} \right) \frac{(\beta-\beta')^2}{|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3}. \tag{5.22}$$

So far we have obtained the approximations of S, T and K in (3.37). Furthermore, we need to discretize (3.36) to get an approximation of $C_{\mathbf{n}}$ which will serve a corrector term to balance off the error generated by the nonsingular kernel $\hat{\mathbf{K}}$ in (3.32). We have $C_{\mathbf{n}} = 0$ in the continuous problem, but in the discrete problem this is not the case due to discretization errors. As before, using the same spectral schemes to discretize $\mathcal{H}_l$ and $\mathcal{G}_{ij}$, and using the standard point vortex method to discretize $\mathbf{K}[\mathbf{X}]\mathbf{N}$, we obtain the following approximation of $C_{\mathbf{n}}$:

$$
\begin{aligned}
C^h_{\mathbf{n}} =& \frac{\boldsymbol{\eta}_h(\boldsymbol{\alpha})}{|\mathbf{N}_h(\boldsymbol{\alpha})|} \cdot \int_h \nabla_{\mathbf{x}} G(\mathbf{X}_h(\boldsymbol{\alpha}) - \mathbf{X}'_h(\boldsymbol{\alpha})) \times \mathbf{N}_h(\boldsymbol{\alpha}')d\boldsymbol{\alpha}' \\
=& -\frac{\boldsymbol{\eta}_h}{4\pi|\mathbf{N}_h|} \cdot \int_h \frac{\mathbf{X}_h - \mathbf{X}'_h}{\left|\mathbf{X}_h - \mathbf{X}'_h\right|^3} \times \mathbf{N}_h(\boldsymbol{\alpha})d\boldsymbol{\alpha}' \\
=& -\frac{\boldsymbol{\eta}_h}{2|\mathbf{N}_h|} \cdot \left(\mathcal{H}^s_{1h}\left(\frac{\mathbf{X}_{\alpha h}}{E_h^{\frac{3}{2}}} \times \mathbf{N}_h\right) + \mathcal{H}^s_{2h}\left(\frac{\mathbf{X}_{\beta h}}{E_h^{\frac{3}{2}}} \times \mathbf{N}_h\right)\right) \\
& - \frac{\boldsymbol{\eta}_h}{|\mathbf{N}_h|} \cdot \mathcal{G}^s_{11,h}\left(\left(\frac{D_{1h}\mathbf{X}_{\alpha h}}{2E_h^{\frac{3}{2}}} - \frac{3\mathbf{X}_{\alpha h}D_{1h}E_h}{4E_h^{\frac{3}{2}}}\right) \times \mathbf{N}_h\right) \\
& - \frac{\boldsymbol{\eta}_h}{|\mathbf{N}_h|} \cdot \mathcal{G}^s_{12,h}\left(\left(\frac{D_{1h}\mathbf{X}_{\beta h}}{2E_h^{\frac{3}{2}}} - \frac{3\left(\mathbf{X}_{\beta h}D_{1h}E_h + \mathbf{X}_{\alpha h}D_{2h}E_h\right)}{4E_h^{\frac{3}{2}}}\right) \times \mathbf{N}_h\right) \\
& - \frac{\boldsymbol{\eta}_h}{|\mathbf{N}_h|} \cdot \mathcal{G}^s_{22,h}\left(\left(\frac{\mathbf{X}_{\beta h}}{2E_h^{\frac{3}{2}}} - \frac{3D_{2h}E_hD_{2h}\mathbf{X}_{\beta h}}{4E_h^{\frac{3}{2}}}\right) \times \mathbf{N}_h\right) \\
& + \frac{\boldsymbol{\eta}_h}{|\mathbf{N}_h|} \cdot \mathbf{K}^p_h[\mathbf{X}]\mathbf{N}_h,
\end{aligned}
\tag{5.23}
$$

where in the first two equalities we used $\int_h$ to denote the mixed integration scheme which calculates the singular and weakly singular integrals using spectral schemes without filtering and the remaining nonsingular integrals using the standard point vortex method. As before we use $\int_h^p$ to indicate the standard point vortex integral and the last term is defined similarly to (5.15):

$$
\mathbf{K}^p_h[\mathbf{X}_h]\mathbf{N}_h = \frac{1}{4\pi} P.V. \int_h^p \mathbf{N}_h(\boldsymbol{\alpha}') \times \hat{\mathbf{K}}_h d\boldsymbol{\alpha}'. \tag{5.24}
$$

With (5.12), (5.14), (5.15) and (5.23), we obtain the following approximation of U from (3.37) :

$$
U_h = S_h + T_h + K_h - C^h_{\mathbf{n}}. \tag{5.25}
$$

As mentioned earlier, we treat $\mathbf{X}_h$, $\mathbf{X}_{\alpha h}$, $\mathbf{X}_{\beta h}$ and κ_h as independent variables. Plugging (5.4), (5.5) and (5.6) into (5.12), (5.14), (5.15) and (5.23), we can see the later are all functions of $\mathbf{X}_h$, $\mathbf{X}_{\alpha h}$, $\mathbf{X}_{\beta h}$. Then from (5.25), we can write U_h as a function of $\mathbf{X}_h$, $\mathbf{X}_{\alpha h}$, $\mathbf{X}_{\beta h}$ and κ_h, denoted by

$$
U_h = \mathcal{U}_h\left(\mathbf{X}_h, \mathbf{X}_{\alpha h}, \mathbf{X}_{\beta h}, \kappa_h\right). \tag{5.26}
$$

Here we use the $\mathcal{U}_h$ notation in order to avoid writing down the lengthy formula of U_h in terms of $\mathbf{X}_h$, $\mathbf{X}_{\alpha h}$, $\mathbf{X}_{\beta h}$, and κ_h, but the formula is explicit and computed in our numerical scheme.

We discretize (2.20), (2.21) to calculate the tangential velocities V_{1h}, V_{2h}. Here both are assumed to be of mean zero for uniqueness. we have

$$
V_{1h} = \Delta_h^{-1}\left(D_{1h}\left(\frac{U_h\left(L_h - N_h\right)}{E_h}\right) + D_{2h}\left(\frac{2U_hM_h}{E_h}\right)\right), \tag{5.27}
$$

$$
V_{2h} = \Delta_h^{-1}\left(D_{1h}\left(\frac{2U_hM_h}{E_h}\right) - D_{2h}\left(\frac{U_h\left(L_h - N_h\right)}{E_h}\right)\right), \tag{5.28}
$$

where the discrete inverse Laplace operator Δ_h^{-1} is defined in terms of its discrete Fourier symbol:

$$
(\widetilde{\Delta_h^{-1} f_h})_{\mathbf{k}} = \begin{cases} 0, & |\mathbf{k}| = 0, \\ -\widetilde{f}_{h\mathbf{k}}/|\mathbf{k}|^2, & |\mathbf{k}| \neq 0. \end{cases} \tag{5.29}
$$

Similarly, we can also write V_{1h} and V_{2h} as explicit functions of $\mathbf{X}_h$, $\mathbf{X}_{\alpha h}$, $\mathbf{X}_{\beta h}$, and κ_h. We use the following notations:

$$V_{1h} = \mathcal{V}_{1h}\left(\mathbf{X}_h, \mathbf{X}_{\alpha h}, \mathbf{X}_{\beta h}, \kappa_h\right), \tag{5.30}$$

$$V_{2h} = \mathcal{V}_{2h}\left(\mathbf{X}_h, \mathbf{X}_{\alpha h}, \mathbf{X}_{\beta h}, \kappa_h\right). \tag{5.31}$$

5.2. **Discretized Evolution Equations.** We are almost ready to give the discrete evolution equations. We introduce the notation $\boldsymbol{\mathcal{V}}^0$,

$$\boldsymbol{\mathcal{V}}_h^0\left(\mathbf{X}_h, \mathbf{X}_{\alpha h}, \mathbf{X}_{\beta h}, \kappa_h\right) = \mathcal{U}_h \mathbf{n}_h + \mathcal{V}_{1h}\mathbf{X}_{\alpha h} + \mathcal{V}_{2h}\mathbf{X}_{\beta h}. \tag{5.32}$$

One might reasonably expect the evolution equation for $\frac{d\mathbf{X}_h}{dt}$ to be given by $\boldsymbol{\mathcal{V}}_h^0\left(\mathbf{X}_h, \mathbf{X}_{\alpha h}, \mathbf{X}_{\beta h}, \kappa_h\right)$. Instead, however, we need to be more careful than this to be able to make energy estimates to demonstrate stability.

In studying the continuous version of problems of this kind, as in, for example, [2], exact relationships between the quantities $\mathbf{X}$, $\mathbf{X}_\alpha$, $\mathbf{X}_\beta$, and κ are frequently used when making estimates. For the discretized problem, however, these exact relationships cannot be expected to hold. As we have mentioned previously, part of our resolution of this problem is to treat $\mathbf{X}_h$, $\mathbf{X}_{\alpha h}$, $\mathbf{X}_{\beta h}$, and κ_h as being independent of each other, severing the link between them, and writing their evolution equations in terms of all of these different quantities. When making energy estimates in the continuous problem, the exact relationships allow us to infer the highest regularity of $\mathbf{X}$ from the regularity of κ : the surface $\mathbf{X}$ is two derivatives smoother than κ. Obviously we cannot do this for the discretized problem, as we do not assume that, at positive times, κ_h and $\mathbf{X}_h$ have a precise relationship. Thus, we need to deal with the question of the regularity of the discretized surface in a different way than in the continuous problem.

In Section 8 below, we will study the stability of the method by making energy estimates; the energy functional is given in (8.36). As can be seen there, the energy includes the L_h^2 norm of $\mathbf{X}_h$. We thus must ensure that the right-hand sides of our discretized evolution equations do not contain terms which require derivatives of $\mathbf{X}_h$ to estimate. Similarly, the energy includes only the L_h^2 norms of $\mathbf{X}_{\alpha h}$ and $\mathbf{X}_{\beta h}$, so the right-hand sides of the discretized evolution equations also should not contain terms which require higher derivatives of $\mathbf{X}_{\alpha h}$ or $\mathbf{X}_{\beta h}$ when making estimates.

To ensure the requisite boundedness properties, we employ a substitution rule. Recall that in the continuous problem, we have (2.8) because of the isothermal parametrization. Suggested by this and the periodicity assumption on the interface, in the numerical scheme, we will make use of a version of the following substitution:

$$\begin{aligned} &\mathbf{X} = \Delta^{-1}\left(2\kappa\mathbf{N}\right) + (\boldsymbol{\alpha}, \boldsymbol{\beta}, 0), && \mathbf{N} = \mathbf{X}_\alpha \times \mathbf{X}_\beta, \\ &\mathbf{X}_\alpha = \partial_\alpha \Delta^{-1}\left(2\kappa\mathbf{N}\right) + (1,0,0), && \mathbf{X}_\beta = \partial_\beta \Delta^{-1}\left(2\kappa\mathbf{N}\right) + (0,1,0). \end{aligned} \tag{5.33}$$

We thus define an operator $\mathcal{M}$ that acts on functions of $\mathbf{X}_h$, $\mathbf{X}_{\alpha h}$ and $\mathbf{X}_{\beta h}$ by using a version of (5.33) to replace these three variables. More precisely, for a function $\Phi\left(\mathbf{X}_h, \mathbf{X}_{\alpha h}, \mathbf{X}_{\beta h}, \kappa_h\right)$, we have

$$\begin{aligned} &\mathcal{M}\Phi\left(\mathbf{X}_h, \mathbf{X}_{\alpha h}, \mathbf{X}_{\beta h}, \kappa_h\right) \\ =&\Phi\left(\Delta_h^{-1}\left(2\kappa_h\mathbf{N}_h\right) + (\boldsymbol{\alpha}_h, \boldsymbol{\beta}_h, 0), D_{1h}\Delta_h^{-1}\left(2\kappa_h\mathbf{N}_h\right) + (1,0,0), D_{2h}\Delta_h^{-1}\left(2\kappa_h\mathbf{N}_h\right) + (0,1,0), \kappa_h\right), \end{aligned} \tag{5.34}$$

where $\mathbf{N}_h = \mathbf{X}_{\alpha h} \times \mathbf{X}_{\beta h}$. We will need, in fact, to apply the operator $\mathcal{M}$ multiple times in some cases. We now give an example of what this means. Take, for example,

$$\Phi(\mathbf{X}_h) = D_{1h}^3 \mathbf{X}_h.$$

Even though this is an artificial example, we mention that for such a function Φ, its norm cannot be estimated using only $\|\mathbf{X}_h\|_{L_h^2}$, and this is the reason for making such a substitution. We first compute $\mathcal{M}\Phi(\mathbf{X}_h)$, and then we will compute $\mathcal{M}^2\Phi(\mathbf{X}_h)$. Right away, we have

$$\mathcal{M}\Phi(\mathbf{X}_h) = D_{1h}^3\left(\Delta_h^{-1}(2\kappa_h\mathbf{N}_h) + (\alpha_h,\beta_h,0)\right) = D_{1h}^3\left(\Delta_h^{-1}(2\kappa_h(\mathbf{X}_{\alpha h}\times\mathbf{X}_{\beta h}))\right). \tag{5.35}$$

We notice that the right-hand side of (5.35) cannot be estimated by a norm of κ_h and $\|\mathbf{X}_{\alpha h}\|_{L_h^2}$ and $\|\mathbf{X}_{\beta h}\|_{L_h^2}$; again, even though this is only an artificial example, this is the reason for making another substitution (i.e., for continuing on to compute $\mathcal{M}^2\Phi$). Continuing on, we find the following:

$$\mathcal{M}^2\Phi(\mathbf{X}_h) = D_{1h}^3\left(\Delta_h^{-1}\left(2\kappa_h\left(\left(D_{1h}\Delta_h^{-1}(2\kappa_h\mathbf{N}_h) + (1,0,0)\right)\times\left(D_{2h}\Delta_h^{-1}(2\kappa_h\mathbf{N}_h) + (0,1,0)\right)\right)\right)\right), \tag{5.36}$$

where as before, we have $\mathbf{N}_h = \mathbf{X}_{\alpha h}\times\mathbf{X}_{\beta h}$. Notice that on the right-hand side of (5.36), we have four factors of D_{1h}, and all occurences of $\mathbf{X}_{\alpha h}$ or $\mathbf{X}_{\beta h}$ occur inside two instances of Δ_h^{-1}; thus, we do not require derivatives of $\mathbf{X}_{\alpha h}$ or $\mathbf{X}_{\beta h}$ to estimate this. In this example, then, we can estimate $\mathcal{M}^2\Phi(\mathbf{X}_h)$ using one derivative of κ_h and zero derivatives of $\mathbf{X}_{\alpha h}$ and $\mathbf{X}_{\beta h}$. This completes the example, and we return now to the task of defining the discretized evolution equations.

We apply the operator $\mathcal{M}$ once to define the evolution equation for $\mathbf{X}_h$:

$$\frac{d\mathbf{X}_h}{dt} = \boldsymbol{\mathcal{V}}_h\left(\mathbf{X}_h,\mathbf{X}_{\alpha h},\mathbf{X}_{\beta h},\kappa_h\right) := \mathcal{M}\boldsymbol{\mathcal{V}}_h^0. \tag{5.37}$$

Similarly, applying $\mathcal{M}$ again, we form the evolution equations for $\mathbf{X}_{\alpha h}$ and $\mathbf{X}_{\beta h}$:

$$\frac{d\mathbf{X}_{\alpha h}}{dt} = \Psi_{1h}\left(\mathbf{X}_h,\mathbf{X}_{\alpha h},\mathbf{X}_{\beta h},\kappa_h\right), \tag{5.38}$$

$$\frac{d\mathbf{X}_{\beta h}}{dt} = \Psi_{2h}\left(\mathbf{X}_h,\mathbf{X}_{\alpha h},\mathbf{X}_{\beta h},\kappa_h\right), \tag{5.39}$$

where

$$\Psi_{1h}\left(\mathbf{X}_h,\mathbf{X}_{\alpha h},\mathbf{X}_{\beta h},\kappa_h\right) = \mathcal{M}D_{1h}\boldsymbol{\mathcal{V}}_h, \qquad \Psi_{2h}\left(\mathbf{X}_h,\mathbf{X}_{\alpha h},\mathbf{X}_{\beta h},\kappa_h\right) = \mathcal{M}D_{2h}\boldsymbol{\mathcal{V}}_h. \tag{5.40}$$

We discretize (4.12) to obtain the evolution equation for κ_h. We use the notations D_{1h}, D_{2h}, E_h, L_h, M_h, N_h, U_h, V_{1h}, V_{2h}, T_h, K_h, $C_{\mathbf{n}}^h$ to denote the same discrete terms as before. The approximation of R is obtained by discretizing (4.10), but before doing so, we will rewrite the commutators which appear there. The product rule and the identity $\Lambda\partial_\alpha = \mathcal{H}_1\Delta$ yield the following:

$$\begin{aligned}\left[\Lambda\partial_\alpha,\frac{1}{\sqrt{E}}\right]\kappa_\alpha &= \mathcal{H}_1\left(\frac{\Delta\kappa_\alpha}{\sqrt{E}}\right) - \frac{1}{\sqrt{E}}\mathcal{H}_1\Delta\kappa_\alpha + 2\mathcal{H}_1\left(\nabla\kappa_\alpha\cdot\nabla\left(\frac{1}{\sqrt{E}}\right)\right) + \mathcal{H}_1\left(\kappa_\alpha\Delta\left(\frac{1}{\sqrt{E}}\right)\right)\\ &= \left[\mathcal{H}_1,\frac{1}{\sqrt{E}}\right]\kappa_\alpha + 2\mathcal{H}_1\left(\nabla\kappa_\alpha\cdot\nabla\left(\frac{1}{\sqrt{E}}\right)\right) + \mathcal{H}_1\left(\kappa_\alpha\Delta\left(\frac{1}{\sqrt{E}}\right)\right).\end{aligned}$$

Similarly, we have

$$\left[\Lambda\partial_\beta,\frac{1}{\sqrt{E}}\right]\kappa_\beta = \left[\mathcal{H}_2,\frac{1}{\sqrt{E}}\right]\Delta\kappa_\beta + 2\mathcal{H}_2\left(\nabla\kappa_\beta\cdot\nabla\left(\frac{1}{\sqrt{E}}\right)\right) + \mathcal{H}_2\left(\kappa_\beta\Delta\left(\frac{1}{\sqrt{E}}\right)\right).$$

Using these formulas for the commutators, we can express the discretized version of R as

(5.41)

$$R_h = \frac{B}{2}\left[\mathcal{H}^p_{1h}, \frac{1}{\sqrt{E_h}}\right]\Delta_h D_{1h}\kappa_h + B\mathcal{H}^s_{1h}\left(\nabla_h D_{1h}\kappa_h \cdot \nabla_h\left(\frac{1}{\sqrt{E_h}}\right)\right) + \frac{B}{2}\mathcal{H}^s_{1h}\left((D_{1h}\kappa_h)\Delta_h\left(\frac{1}{\sqrt{E_h}}\right)\right)$$
$$+ \frac{B}{2}\left[\mathcal{H}^p_{2h}, \frac{1}{\sqrt{E_h}}\right]\Delta_h D_{2h}\kappa_h + B\mathcal{H}^s_{2h}\left(\nabla_h D_{2h}\kappa_h \cdot \nabla_h\left(\frac{1}{\sqrt{E_h}}\right)\right) + \frac{B}{2}\mathcal{H}^s_{2h}\left((D_{2h}\kappa_h)\Delta_h\left(\frac{1}{\sqrt{E_h}}\right)\right)$$
$$- \Delta_h\left(\frac{D_{1h}E_h}{2E_h}\mathcal{G}^s_{11,h} + \frac{D_{1h}E_h}{4E_h}\mathcal{G}^s_{22,h} + \frac{D_{2h}E_h}{4E_h}\mathcal{G}^s_{12,h}\right)\frac{r_{1h}}{\sqrt{E_h}}$$
$$- \Delta_h\left(\frac{D_{2h}E_h}{4E_h}\mathcal{G}^s_{11,h} + \frac{D_{2h}E_h}{2E_h}\mathcal{G}^s_{22,h} + \frac{D_{1h}E_h}{4E_h}\mathcal{G}^s_{12,h}\right)\frac{r_{2h}}{\sqrt{E_h}}.$$

Then with (5.41), (5.14), (5.23), (5.15), (5.4), (5.26), (5.30) and (5.31), the right-hand side of (4.12) can be discretized as

$$\begin{aligned}\Gamma^0_h :=& -\frac{B}{4E_h^{3/2}}\Lambda^3_h\kappa_h + \frac{1}{2E_h}\left(R_h + \Delta_h T_h + \Delta_h K_h - \Delta_h C^h_n\right)\\ &+ \frac{\kappa_h}{2E_h}\left(2U_h L_h - 2(D_{1h}V_{1h})E_h - V_{1h}(D_{1h}E_h) - V_{2h}D_{2h}E_h\right)\\ &+ \frac{1}{2E_h}\left(D_{1h}\left(V_{1h}L_h + V_{2h}M_h\right) + D_{2h}\left(V_{1h}M_h + V_{2h}N_h\right)\right).\end{aligned} \tag{5.42}$$

Similarly to (5.26), here we write the right hand side of (5.42) as a function of $\mathbf{X}_h$,$\mathbf{X}_{\alpha h}$,$\mathbf{X}_{\beta h}$ and κ_h; that is, we may write $\Gamma^0_h = \Gamma^0_h\left(\mathbf{X}_h, \mathbf{X}_{\alpha h}, \mathbf{X}_{\beta h}, \kappa_h\right)$. We then apply (5.33) six times. The evolution equation for κ_h is then given by the following:

$$\frac{d\kappa_h}{dt} = \Gamma_h\left(\mathbf{X}_h, \mathbf{X}_{\alpha h}, \mathbf{X}_{\beta h}, \kappa_h\right), \tag{5.43}$$

where

$$\Gamma_h\left(\mathbf{X}_h, \mathbf{X}_{\alpha h}, \mathbf{X}_{\beta h}, \kappa_h\right) = \mathcal{M}^6\Gamma^0_h\left(\mathbf{X}_h, \mathbf{X}_{\alpha h}, \mathbf{X}_{\beta h}, \kappa_h\right). \tag{5.44}$$

We close this section by stating the main result in this paper on the convergence properties of our modified point vortex method consisting of the evolution equations (5.37), (5.38), (5.39), (5.43).

Theorem 5.1. *Suppose the model problem is well-posed and has a sufficiently smooth solution* $\mathbf{X}$ *up to time* $T > 0$. *In addition we assume that* $\mathbf{X}$ *is nonsingular and satisfies conditions* (2.22) *and* (2.23). *Then the modified point vortex method described by* (5.37), (5.38), (5.39) *and* (5.43) *is stable and* 3^{rd}*-order accurate. More precisely, there exists a positive number* $h_0(T)$ *such that for all* $0 < h < h_0(T)$, *we have*

$$\|\mathbf{X} - \mathbf{X}_h\|_{L^2_h} \le C(T)h^3, \tag{5.45}$$

where $\|\cdot\|_{L^2_h}$ *is the discrete* l^2 *norm over a period of* $\boldsymbol{\alpha}$, *i.e.,* $\|\mathbf{x}\|^2_{L^2_h} = \sum_{i,j=-N/2+1}^{N/2}|\mathbf{x}_{i,j}|^2 h^2$, *and* $C(T) > 0$ *is a constant that does not depend on* h.

The proof of Theorem 5.1 will be the content of the remaining sections.

6. Consistency of the modified point vortex method

In this section, we prove that the modified point vortex method has an error expansion in odd powers of h and is 3^{rd}-order accurate in $\mathbf{X}$, 2^{nd}-order accurate in $\mathbf{X}_\alpha$, $\mathbf{X}_\beta$, and 1^{st}-order accurate in κ. Our approach is similar to that used in [20] and we only provide a sketch of the proof.

As seen in Section 5, the evolution equations for (5.38) and (5.39) are obtained by taking spectral derivatives D_{1h}, D_{2h} of (5.37), and the evolution equation (5.43) is basically obtained by taking Δ_h of (5.37) except that we reformulated ΔS in the continuous problem. Therefore, we only need to prove the $O(h^3)$ accuracy of equation (5.37). Recall that the tangential velocities V_{1h} and V_{2h} are also spectrally calculated from and of lower order than U_h. Therefore, it suffices to prove the $O(h^3)$ accuracy of the normal velocity approximation U_h. Furthermore, the term S_h in the decomposition (5.25) for U_h is spectrally calculated, so we need only consider the errors of the standard point vortex approximation in T_h, K_h, and the part of $C^h_{\boldsymbol{n}}$ in (5.23) that is approximated by the point vortex method.

Let $-\mathbf{v}(\boldsymbol{\alpha},\boldsymbol{\alpha}')/(4\pi)$ be the intergrand of the part of U_h that is computed from the standard point vortex approximation. Then from (2.6), (3.29), (3.30), and (5.24) we have

(6.1)

$$\begin{aligned}\mathbf{v}(\boldsymbol{\alpha},\boldsymbol{\alpha}') =& \mathbf{n}(\boldsymbol{\alpha})\cdot\boldsymbol{\eta}(\boldsymbol{\alpha}')\times\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|^3}+\frac{r_1(\boldsymbol{\alpha}')(\alpha-\alpha')}{E^{\frac{1}{2}}(\boldsymbol{\alpha}')\,|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3}+\frac{r_2(\boldsymbol{\alpha}')(\beta-\beta')}{E^{\frac{1}{2}}(\boldsymbol{\alpha}')\,|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3}\\ &-\frac{\left(\frac{1}{2}r_1(\boldsymbol{\alpha}')E_\alpha(\boldsymbol{\alpha})+\frac{1}{4}r_2(\boldsymbol{\alpha}')E_\beta(\boldsymbol{\alpha})\right)(\alpha-\alpha')^2}{E(\boldsymbol{\alpha})E^{\frac{1}{2}}(\boldsymbol{\alpha}')\,|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3}-\frac{\left(\frac{1}{4}r_1(\boldsymbol{\alpha}')E_\alpha(\boldsymbol{\alpha})+\frac{1}{2}r_2(\boldsymbol{\alpha}')E_\beta(\boldsymbol{\alpha})\right)(\beta-\beta')^2}{E(\boldsymbol{\alpha})E^{\frac{1}{2}}(\boldsymbol{\alpha}')\,|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3}\\ &-\frac{\left(\frac{1}{4}r_1(\boldsymbol{\alpha}')E_\beta(\boldsymbol{\alpha})+\frac{1}{4}r_2(\boldsymbol{\alpha}')E_\alpha(\boldsymbol{\alpha})\right)(\alpha-\alpha')(\beta-\beta')}{E(\boldsymbol{\alpha})E^{\frac{1}{2}}(\boldsymbol{\alpha}')\,|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|^3}+\frac{\eta(\boldsymbol{\alpha})}{|\mathbf{N}(\boldsymbol{\alpha})|}\cdot\mathbf{N}(\boldsymbol{\alpha}')\times\hat{\mathbf{K}}.\end{aligned}$$

Since $\mathbf{v}(\boldsymbol{\alpha},\boldsymbol{\alpha}')$ is the corresponding integrand of the standard point vortex integral of (5.25), it suffices to show the following:

$$P.V.\int\mathbf{v}(\boldsymbol{\alpha},\boldsymbol{\alpha}')d\boldsymbol{\alpha}'-P.V.\int_h^p\mathbf{v}(\boldsymbol{\alpha},\boldsymbol{\alpha}')d\boldsymbol{\alpha}'=C_3h^3+C_5h^5+\cdots+C_{2n+1}h^{2n+1}+\cdots, \tag{6.2}$$

where $\int_h^p$ denotes the standard point vortex approximation and $C_3, C_5,\cdots$ are constants.

This error expansion can be proved using an argument similar to that presented in [20]. Define a smooth cut-off function f_δ such that (i) $f_\delta(|x|)=1$ for $|x|\le\delta/2$; (ii) $f_\delta(|x|)=0$ for $|x|\ge\delta$. Here the constant $\delta>0$ is taken to be small and independent of h. Then decompose $\mathbf{v}$ as follows:

$$\mathbf{v}(\boldsymbol{\alpha},\boldsymbol{\alpha}')=\mathbf{v}(\boldsymbol{\alpha},\boldsymbol{\alpha}')f_\delta(\left|\boldsymbol{\alpha}-\boldsymbol{\alpha}'\right|)+\mathbf{v}(\boldsymbol{\alpha},\boldsymbol{\alpha}')(1-f_\delta(\left|\boldsymbol{\alpha}-\boldsymbol{\alpha}'\right|)). \tag{6.3}$$

As noted in [20], classical error analysis shows that the far-field part under point vortex approximation is spectrally accurate, see, e.g. [16]. More precisely, we have

$$\left|\int\mathbf{v}(\boldsymbol{\alpha},\boldsymbol{\alpha}')(1-f_\delta(\left|\boldsymbol{\alpha}-\boldsymbol{\alpha}'\right|))d\boldsymbol{\alpha}'-\int_h^p\mathbf{v}(\boldsymbol{\alpha},\boldsymbol{\alpha}')(1-f_\delta(\left|\boldsymbol{\alpha}-\boldsymbol{\alpha}'\right|))d\boldsymbol{\alpha}'\right|\le Ch^M, \tag{6.4}$$

where M is the regularity of $\mathbf{X}$.

For the near field part, we Taylor-expand $\mathbf{v}(\boldsymbol{\alpha},\boldsymbol{\alpha}')$ in terms of $\boldsymbol{\alpha}-\boldsymbol{\alpha}'$. This is done by Taylor-expanding $\mathbf{n}$ and $\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|^3}$ in (6.1). We get

$$\mathbf{v}(\boldsymbol{\alpha},\boldsymbol{\alpha}')=\mathbf{m}_{-2}(\boldsymbol{\alpha}'-\boldsymbol{\alpha})+\mathbf{m}_{-1}(\boldsymbol{\alpha}'-\boldsymbol{\alpha})+\mathbf{m}_0(\boldsymbol{\alpha}'-\boldsymbol{\alpha})+\mathbf{m}_1(\boldsymbol{\alpha}'-\boldsymbol{\alpha})+\cdots. \tag{6.5}$$

where $\boldsymbol{m}_l(\boldsymbol{\alpha})$ are homogeneous functions of degree l.

For any even number l, $\mathbf{m}_l$ is an odd function. Therefore, $\mathbf{m}_l(\boldsymbol{\alpha}'-\boldsymbol{\alpha})f_\delta(|\boldsymbol{\alpha}-\boldsymbol{\alpha}'|)$ is also an odd function of $\boldsymbol{\alpha}-\boldsymbol{\alpha}'$ and hence it does not contribute to neither the continuous nor the discrete principle integral, that is,

$$\int\mathbf{m}_l(\boldsymbol{\alpha},\boldsymbol{\alpha}')f_\delta(\left|\boldsymbol{\alpha}-\boldsymbol{\alpha}'\right|)d\boldsymbol{\alpha}'-\int_h^p\mathbf{m}_l(\boldsymbol{\alpha},\boldsymbol{\alpha}')f_\delta(\left|\boldsymbol{\alpha}-\boldsymbol{\alpha}'\right|)d\boldsymbol{\alpha}'=0. \tag{6.6}$$

The Taylor expansion also shows that, crucially, $\mathbf{m}_{-1}(\boldsymbol{\alpha}' - \boldsymbol{\alpha}) = 0$. As a result, the first term in (6.5) that contributes to the error of the point vortex approximation of the near field is the $\mathbf{m}_1$ term. It is proved in [20] that for any integer $l \geq 1$, we have

$$\int \mathbf{m}_l(\boldsymbol{\alpha}, \boldsymbol{\alpha}') f_\delta(|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|) d\boldsymbol{\alpha}' - \int_h^p \mathbf{m}_l(\boldsymbol{\alpha}, \boldsymbol{\alpha}') f_\delta(|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|) d\boldsymbol{\alpha}' = C_{l+2} h^{l+2} + O(h^M), \tag{6.7}$$

where M again denotes the regularity of $\mathbf{X}$. This completes the consistency proof.

7. Boundedness properties of the discretized integral operators

In this section we derive boundedness properties of the involved discrete integral operators and commutators in (5.26). The results are largely analogues of the properties for the corresponding continuous integrals presented in [4]. Some related arguments and results from [11, 20] are also adopted here.

First we introduce some notations. For some constant $s \geq 0$, as an analogue of the continuous Sobolev space H^s, we introduce the discrete Sobolev space $H_h^s \subset L_h^2$. For any $f_h \in L_h^2$, define its discrete Sobolev space norm as

$$\|f_h\|_{H_h^s} = \|(1 + \Lambda_h^s) f_h\|_{L_h^2} . \tag{7.1}$$

The collection of all $f_h \in L_h^2$ such that $\|f_h\|_{H_h^s}$ is bounded is called the discrete Sobolev space of order s and denoted by H_h^s throughout this paper. In particular, when $s = 0$, $H_h^s = L_h^2$. For the discrete Sobolev norm $\|\cdot\|_{H_h^s}$, we have the following result:

Lemma 7.1. *Suppose that f is a 2π-doubly-periodic and sufficiently smooth function on $\mathbb{R}^2$. Consider a discrete 2π-doubly-periodic function $g_h \in H_h^s$ for some constant $s \geq 0$. Then the product function $f g_h \in H_h^s$. More precisely, we have*

$$\|f g_h\|_{H_h^s} \leq C \|g_h\|_{H_h^s} , \tag{7.2}$$

for some constant $C > 0$ that only depends on f.

Proof. This result can be proved by calculating the H_h^s norm from the definition (7.1), applying the Plancherel Theorem and writing $\widehat{f g_h}$ as a circular convolution of $\widehat{f}$ and $\widehat{g_h}$. By assumption f is sufficiently smooth and periodic, hence we have $\left|\widehat{f_{\mathbf{k}}}\right| \leq C |\mathbf{k}|^{-q}$, for some constant $C > 0$ that only depends on f and any $q > 0$. This result is useful in the proof and the rest is skipped here. ☐

Consider a function $f_h \in H_h^s$ for some constant $s \geq 0$. For another constant $m \leq s$, we use $A_m(f_h)$ to denote a term such that for any constant k, $0 \leq k \leq s - m$, we have

$$\|A_m(f_h)\|_{H_h^k} \leq C \|f_h\|_{H_h^{k+m}} , \tag{7.3}$$

where $C > 0$ denote constants that do not depend on h. If g_h is such a term, we write $g_h \in A_m(f_h)$. In particular, when the $A_m(f_h)$ term is of the form $\mathcal{I}_h f_h$, where $\mathcal{I}_h$ is an operator acting on L_h^2, we say that $\mathcal{I}_h$ is A_m or $\mathcal{I}_h \in A_m$, for the constant s. We often do not mention s as long as there is no ambiguity. From this definition, we see that for all $m \leq n \leq s$, we have $A_m \in A_n$ and $A_m A_n \in A_{m+n}$, $A_n A_m \in A_{m+n}$. Also, we often write $A_m(f_h) + A_m(g_h)$ as $A_m(f_h, g_h)$.

Now we are ready to state some boundedness properties of the discrete operators in our numerical scheme. For the spectrally discretized derivatives and integral operators described in Section 5, we have

Lemma 7.2. *For any 2π-doubly-periodic function $f_h \in H_h^s$, where $s \geq 1$ is some constant, the spectrally discretized operators D_{lh}, $l = 1, 2$, Λ_h are A_1, $\mathcal{H}_{lh}^s$, $l = 1, 2$ are A_0, $G_{ij,h}^s$, $(i, j) = (1, 1), (1, 2), (2, 2)$ are A_{-1}, and Δ_h^{-1} is A_{-2}.*

Proof. The results are immediate from the definition of A_m and the discrete Fourier symbols of these operators. ☐

For the point vortex integral operators $\mathcal{H}^p_{lh}$, $l = 1, 2$ and $\mathcal{G}^p_{ij,h}$, $(i, j) = (1, 1), (1, 2), (2, 2)$, we have the following boundedness properties:

Lemma 7.3. *For any 2π-doubly-periodic function $f_h \in H^s_h$, where $s \geq 0$ is some constant, the standard point vortex approximations of the Riesz transforms, $\mathcal{H}^p_{lh}$, $l = 1, 2$ are A_0, and the point vortex integral operators $G^p_{ij,h}$, $(i, j) = (1, 1), (1, 2), (2, 2)$ are A_{-1}.*

Proof. We will only prove that $\mathcal{H}^p_{lh}$, $l = 1, 2$ are A_0, since the result for $G^p_{ij,h}$, $(i, j) = (1, 1), (1, 2), (2, 2)$ can be proved similarly. For any any 2π-double-periodic function $f_h \in H^s_h$, we have

$$\mathcal{H}^p_{lh} f_h(\boldsymbol{\alpha}) = \frac{1}{2\pi} P.V. \int_h^p f_h(\boldsymbol{\alpha}') \frac{\alpha_l - \alpha'_l}{|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|^3} d\boldsymbol{\alpha}' = \frac{1}{2\pi} P.V. \int_h f_h(\boldsymbol{\alpha}') g_l(\boldsymbol{\alpha} - \boldsymbol{\alpha}') d\boldsymbol{\alpha}', \tag{7.4}$$

where

$$g_l(\boldsymbol{\alpha}) = \begin{cases} 0, & |\boldsymbol{\alpha}| = 0, \\ \alpha_l / |\boldsymbol{\alpha}|^3, & |\boldsymbol{\alpha}| > 0. \end{cases} \tag{7.5}$$

We see that $\mathcal{H}^p_{lh}$, $l = 1, 2$ are discrete convolution operators with kernels $\frac{1}{2\pi} g_l$. The discrete Fourier symbols of $\mathcal{H}^p_{lh}$, $l = 1, 2$ are presented in [20]:

$$\left(\widetilde{\mathcal{H}^p_{lh} f_h}\right)_{\mathbf{k}} = \begin{cases} 0, & |\mathbf{k}| = 0, \\ -\frac{ik_l}{|\boldsymbol{k}|} b_l(\mathbf{k}h) \tilde{f}_{\mathbf{k}}, , & |\mathbf{k}| > 0, \end{cases} \tag{7.6}$$

where $l = 1, 2$, $\mathbf{k} = (k_1, k_2)$, $k_1, k_2 = -\frac{N}{2} + 1, \cdots, \frac{N}{2}$ and

$$b_1(\mathbf{k}h) = \frac{1}{2\pi} P.V. \sum_{\mathbf{j} \neq 0} \frac{j_1 \sin(j_1 k_1 h) \cos(j_2 k_2 h)}{(j_1^2 + j_2^2)^{3/2}}, \tag{7.7}$$

$$b_2(\mathbf{k}h) = \frac{1}{2\pi} P.V. \sum_{\mathbf{j} \neq 0} \frac{j_2 \sin(j_2 k_2 h) \cos(j_1 k_1 h)}{(j_1^2 + j_2^2)^{3/2}}. \tag{7.8}$$

It can be shown that the above infinite series converge. Hence, from the Parseval's theorem for discrete Fourier transforms, we see that $\mathcal{H}^p_{lh}$, $l = 1, 2$ are A_0. ☐

For a sufficiently smooth 2π-periodic function f on $\mathbb{R}^2$, we consider the commutator $[\mathcal{H}^p_{lh}, f](\cdot)$. It has the following boundedness properties:

Lemma 7.4. *Suppose that f is a sufficiently smooth 2π-doubly-periodic function on $\mathbb{R}^2$. Then when acting on any 2π-doubly-periodic function $g_h \in H^s_h$, where $s \geq 0$ is some constant, the commutator operators $[\mathcal{H}^p_{lh}, f](\cdot)$, $l = 1, 2$ are A_{-1}.*

Proof. The is the discrete version of the result presented by Theorem 6.6 in [4]. For any 2π-doubly-periodic function $g_h \in H^s_h$, we have

$$[\mathcal{H}^p_{lh}, f] g_h = P.V. \int_h^p \left(f(\boldsymbol{\alpha}') - f(\boldsymbol{\alpha})\right) \frac{(\alpha_l - \alpha'_l)}{|\boldsymbol{\alpha} - \boldsymbol{\alpha}|^3} g_h(\boldsymbol{\alpha}') d\boldsymbol{\alpha}'. \tag{7.9}$$

We treat $\boldsymbol{\alpha}$ as a perturbation of $\boldsymbol{\alpha}'$ and make a Taylor expansion of $f(\boldsymbol{\alpha}) - f(\boldsymbol{\alpha}')$ in terms of $\boldsymbol{\alpha} - \boldsymbol{\alpha}'$ with remainder:

$$[\mathcal{H}^p_{lh}, f] g_h = P.V. \int_h^p \left(\sum_{|\mathbf{m}|=1} D^{\mathbf{m}} f(\boldsymbol{\alpha}')(\boldsymbol{\alpha} - \boldsymbol{\alpha}')^{\mathbf{m}} + \sum_{|\mathbf{m}|=2} R_{\mathbf{m}}(\boldsymbol{\alpha}')(\boldsymbol{\alpha} - \boldsymbol{\alpha}')^{\mathbf{m}} \right) \frac{(\alpha_l - \alpha'_l)}{|\boldsymbol{\alpha} - \boldsymbol{\alpha}|^3} g_h(\boldsymbol{\alpha}') d\boldsymbol{\alpha}', \tag{7.10}$$

where $\mathbf{m} = (m_1, m_2)$ is a two-dimensional multi-index with m_1, m_2 being nonnegative integers and $R_{\mathbf{m}}$ are the reminders from the Taylor expansion. Note that all the terms $D^{\mathbf{m}} f$, $|\mathbf{m}| = 1$ and $R_{\mathbf{m}}$, $|\mathbf{m}| = 2$ are sufficiently smooth and 2π-periodic functions, and all these discrete integral operators with kernels $(\boldsymbol{\alpha} - \boldsymbol{\alpha}')^{\mathbf{m}} \frac{(\alpha_l - \alpha_l')}{|\boldsymbol{\alpha} - \boldsymbol{\alpha}|^3}$ are discrete convolution operators that are A_{-1} as seen from their discrete Fourier symbols similar to that in the proof for Theorem 7.3. Hence, we conclude that $[\mathcal{H}_{lh}^p, f] = A_{-1}$ using Lemma 7.1. □

In particular, when $f = \mathbf{n}$ in Lemma 7.4, we have

Lemma 7.5. *For any 2π-double-periodic function $g_h \in H_h^s$, where $s \geq 0$ is a constant, the operators $\mathbf{n} \cdot [\mathcal{H}_{lh}^p, \mathbf{n}](\cdot)$, $l = 1, 2$ are A_{-2}.*

Proof. This is the discrete version of the result presented by Theorem 6.7 in [4]. We apply the same argument used in the proof of Lemma 7.4 and notice that in the Taylor expansion of $\mathbf{n}(\boldsymbol{\alpha}) - \mathbf{n}(\boldsymbol{\alpha}')$, the first term vanishes because $\mathbf{n} \cdot \partial_\alpha \mathbf{n} = 0$. The rest of the proof is similar and the details are skipped. □

Regarding the boundedness properties of the commutators $[\mathcal{G}_{ij,h}^p, f](\cdot)$, we have the following result:

Lemma 7.6. *Suppose that f is a sufficiently smooth 2π-doubly-periodic function on $\mathbb{R}^2$. Then for any 2π-double-periodic function $g_h \in H_h^s$, where $s \geq 0$ is some constant, the commutator operators $[\mathcal{G}_{ij,h}^p, f](\cdot)$, $(i, j) = (1, 1), (1, 2), (2, 2)$ are A_{-2}.*

Proof. This is the discretized version of the result presented by Theorem 6.6 in [4] and the proof is similar to that of the previous Lemma 7.4. The key is to Taylor-expand $f(\boldsymbol{\alpha}') - f(\boldsymbol{\alpha})$ and consider the kernels of the related discrete convolution operators. We skip the details. □

At last, we consider the boundedness properties of the remaining operator $\mathbf{K}_h^p[\mathbf{X}](\cdot)$.

Lemma 7.7. *Acting on any 2π-doubly-periodic function $f_h \in H_h^s$, for some constant $s \geq 0$, the discrete integral operator $\mathbf{K}_h^p[\mathbf{X}](\cdot)$ is A_{-2}.*

Before giving the proof, we mention that it would in fact be sufficient for the operator $\mathbf{K}_h^p[\mathbf{X}]\cdot$ to be $A_{-3/2}$, and in fact, in [4], we used the corresponding continuous operator to gain $3/2$ of a derivative. Nonetheless, it actually gains two derivatives.

Proof. This is the discrete version of the result presented by Lemma 6.1 in [4]. We do a Laurent expansion of $\frac{\mathbf{X} - \mathbf{X}'}{|\mathbf{X} - \mathbf{X}'|^3}$ in terms of $(\boldsymbol{\alpha} - \boldsymbol{\alpha}')^{\mathbf{m}}$, where $\mathbf{m} = (m_1, m_2)$, $m_1, m_2 \geq -1$ are integers and $|\mathbf{m}| = m_1 + m_2 \geq -1$. The first two terms are $\hat{\mathbf{H}}$ and $\hat{\mathbf{G}}$. Hence the kernel $\hat{\mathbf{K}}$ consist of terms like $R_{\mathbf{m}}(\boldsymbol{\alpha} - \boldsymbol{\alpha}')^{\mathbf{m}}$, $|\mathbf{m}| \geq 1$. Then similar to the previous proofs, applying Lemma 7.1, we obtain the conclusion. □

8. Proof of the main theorem

In this section, we perform stability analysis of the numerical method presented in Section 5, and this will complete the proof of the main convergence result (Theorem 5.1). For a discretized operator $\mathcal{I}_h$ acting on L_h^2, we use either $Er(\mathcal{I})$ or the dotted notation $\dot{\mathcal{I}}$ to denote the related error on $\mathcal{I}(\mathbf{X})$:

$$Er(\mathcal{I}) = \dot{\mathcal{I}} = \mathcal{I}_h(\mathbf{X}) - \mathcal{I}_h(\mathbf{X}_h). \tag{8.1}$$

For example, for the normal velocity we have $Er(U) = \dot{U} = U - U_h$. With the notation, what we need to prove is the following: When h is sufficiently small, on $[0,T]$ we have

$$\left\|\dot{\mathbf{X}}\right\|_{L_h^2} \leq Ch^3, \tag{8.2}$$

where $C > 0$ is a uniform constant on $[0,T]$ that does not depend h. The proof is based on an extension argument. First we assume that for some small T^*, $0 < T^* < T$ and some sufficiently large constant $s > 0$, we have

$$\left\|\dot{\mathbf{X}}\right\|_{L_h^2} \leq Ch^s, \tag{8.3}$$

on $[0,T^*]$, where $C > 0$ is a uniform constant that does not depend on h. (An appropriate value of s will be chosen at the end of the argument.) Notice that the discrete systems are continuous in time t, and at $t = 0$, we start with the exact solution of the interface, that is, $\mathbf{X}(\boldsymbol{\alpha}_h, 0) = \mathbf{X}_h(\boldsymbol{\alpha}_h, 0)$. Hence there exists a $T^* > 0$ such that (8.3) holds for us to start with. Then we use an energy estimate to show that the time interval on which (8.3) holds can be extended to $[0,T]$. We therefore need to properly construct an energy and estimate the related error terms.

Consider two discrete operators $\mathcal{I}_{1h}, \mathcal{I}_{2h}$. The error for the product term $\mathcal{I}_1(\mathbf{X}) \cdot \mathcal{I}_2(\mathbf{X})$ is

$$Er(\mathcal{I}_1 \cdot \mathcal{I}_2) = \dot{\mathcal{I}}_1 \cdot \mathcal{I}_2 + \mathcal{I}_1 \cdot \dot{\mathcal{I}}_2 - \dot{\mathcal{I}}_1 \cdot \dot{\mathcal{I}}_2. \tag{8.4}$$

Suppose that $\mathcal{I}_1 = A_m$ and $\mathcal{I}_2 = A_n$, for some constants $m, n \geq 0$. From the assumption (8.3), we get $\left\|\dot{\mathcal{I}}_1\right\|_{L_h^2} \leq Ch^{s-m}$, $\left\|\dot{\mathcal{I}}_2\right\|_{L_h^2} \leq Ch^{s-n}$, for some uniform constant $C > 0$ on $[0,T^*]$ that does not depend on h. Then we can conclude that $\left\|\dot{\mathcal{I}}_1\right\|_{L_h^\infty} \leq C' h^{s-m-1}$, $\left\|\dot{\mathcal{I}}_2\right\|_{L_h^\infty} \leq C' h^{s-n-1}$ on $[0,T^*]$ for some constant $C' \geq 0$ that only depends on C. Hence $\dot{\mathcal{I}}_1 \cdot \dot{\mathcal{I}}_2 = h^{s-n-1} A_m$ and $\dot{\mathcal{I}}_1 \cdot \dot{\mathcal{I}}_2 = h^{s-m-1} A_n$. Given the assumption that s is sufficiently large, we know that this nonlinear error term $\dot{\mathcal{I}}_1 \cdot \dot{\mathcal{I}}_2$ can be merged into either $\dot{\mathcal{I}}_1$ or $\dot{\mathcal{I}}_2$ in (8.4). For this reason, we neglect the nonlinear errors such $\dot{\mathcal{I}}_1 \cdot \dot{\mathcal{I}}_2$ and often skip them in the equations without further explanation.

We have proved in Section 6 the consistency of our modified point vortex method.

$$\mathbf{X}_t = \boldsymbol{\mathcal{V}}_h \left(\mathbf{X}, \mathbf{X}_\alpha, \mathbf{X}_\beta, \kappa\right) + O(h^3), \tag{8.5}$$

$$\mathbf{X}_{\alpha t} = \Psi_{1h} \left(\mathbf{X}, \mathbf{X}_\alpha, \mathbf{X}_\beta, \kappa\right) + O(h^2), \tag{8.6}$$

$$\mathbf{X}_{\beta t} = \Psi_{2h} \left(\mathbf{X}, \mathbf{X}_\alpha, \mathbf{X}_\beta, \kappa\right) + O(h^2), \tag{8.7}$$

$$\kappa_t = \Gamma_h \left(\mathbf{X}, \mathbf{X}_\alpha, \mathbf{X}_\beta, \kappa\right) + O(h). \tag{8.8}$$

However, in the energy estimate analysis, extending $[0,T^*]$ will require higher accuracy. As shown in [20], this technical difficulty can be resolved using Strang's technique [23]. Recall the assumption that the actual solution $\mathbf{X}$ is sufficiently smooth. Then, for any sufficiently large number $r > 0$, there is an $O(h^3)$ perturbation of the exact solution $\mathbf{X}$, denoted by $\widetilde{\mathbf{X}}$, such that the modified point vortex approximation of $\widetilde{\mathbf{X}}$ is of accuracy $O(h^r)$ (and similar statements hold for $\mathbf{X}_\alpha, \mathbf{X}_\beta$, and κ). More precisely, we have

$$\widetilde{\mathbf{X}}_t = \boldsymbol{\mathcal{V}}_h \left(\widetilde{\mathbf{X}}, \widetilde{\mathbf{X}}_\alpha, \widetilde{\mathbf{X}}_\beta, \widetilde{\kappa}\right) + O(h^r), \tag{8.9}$$

$$\widetilde{\mathbf{X}}_{\alpha t} = \Psi_{1h} \left(\widetilde{\mathbf{X}}, \widetilde{\mathbf{X}}_\alpha, \widetilde{\mathbf{X}}_\beta, \widetilde{\kappa}\right) + O(h^{r-1}), \tag{8.10}$$

$$\widetilde{\mathbf{X}}_{\beta t} = \Psi_{2h} \left(\widetilde{\mathbf{X}}, \widetilde{\mathbf{X}}_\alpha, \widetilde{\mathbf{X}}_\beta, \widetilde{\kappa}\right) + O(h^{r-1}), \tag{8.11}$$

$$\widetilde{\kappa}_t = \Gamma_h \left(\widetilde{\mathbf{X}}, \widetilde{\mathbf{X}}_\alpha, \widetilde{\mathbf{X}}_\beta, \widetilde{\kappa}\right) + O(h^{r-2}). \tag{8.12}$$

Also, $\widetilde{\mathbf{X}}$ is also sufficiently smooth by the way it is constructed. Then, to prove the main convergence result (Theorem 5.1), it suffices to show that when h is sufficiently small, on $[0,T]$ we have

$$\left\|\widetilde{\mathbf{X}}-\mathbf{X}_h\right\|_{L_h^2}\leq C(T)h^3, \tag{8.13}$$

$$\left\|\widetilde{\mathbf{X}}_\alpha-\mathbf{X}_{\alpha h}\right\|_{L_h^2}\leq C(T)h^2, \tag{8.14}$$

$$\left\|\widetilde{\mathbf{X}}_\beta-\mathbf{X}_{\beta h}\right\|_{L_h^2}\leq C(T)h^2, \tag{8.15}$$

$$\|\widetilde{\kappa}-\kappa_h\|_{L_h^2}\leq C(T)h, \tag{8.16}$$

where $C(T)>0$ are constants that do not depend h and the tilde notations are used to denote the corresponding terms related to $\widetilde{\mathbf{X}}$. With this in mind, in the remaining part of the paper, we assume that the modified point vortex approximation is sufficiently accurate itself, by which we mean

$$\mathbf{X}_t=\boldsymbol{\mathcal{V}}_h\left(\mathbf{X},\mathbf{X}_\alpha,\mathbf{X}_\beta,\kappa\right)+O(h^r), \tag{8.17}$$

$$\mathbf{X}_{\alpha t}=\Psi_{1h}\left(\mathbf{X},\mathbf{X}_\alpha,\mathbf{X}_\beta,\kappa\right)+O(h^{r-1}), \tag{8.18}$$

$$\mathbf{X}_{\beta t}=\Psi_{2h}\left(\mathbf{X},\mathbf{X}_\alpha,\mathbf{X}_\beta,\kappa\right)+O(h^{r-1}), \tag{8.19}$$

$$\kappa_t=\Gamma_h\left(\mathbf{X},\mathbf{X}_\alpha,\mathbf{X}_\beta,\kappa\right)+O(h^{r-2}), \tag{8.20}$$

where $r>0$ is a sufficiently large constant and the $O(h^r)$, $O(h^{r-1})$, $O(h^{r-2})$ notations are used to denote terms that are bounded by Ch^r, Ch^{r-1}, Ch^{r-2} respectively for some uniform constant $C>0$ on $[0,T]$ that is independent of h.

First, we consider the errors on the velocities $\dot{U}$, $\dot{V}_1$ and $\dot{V}_2$, taking into account the operator $\mathcal{M}$. We recall the definition (5.34) of $\mathcal{M}$, and we consider the error in $\mathcal{M}U$ in the following lemma.

Lemma 8.1. *The error in the normal velocity satisfies* $Er(\mathcal{M}U)=A_1(\dot\kappa)+A_0(\dot{\mathbf{X}},\dot{\mathbf{X}}_\alpha,\dot{\mathbf{X}}_\beta)$.

Proof. From (5.26), we have

$$Er(\mathcal{M}U)=Er(\mathcal{M}S)+Er(\mathcal{M}T)+Er(\mathcal{M}(K-C_{\mathbf{n}}^h)). \tag{8.21}$$

From (5.12), skipping the nonlinear errors (as described above) and applying Lemma 7.1, Lemma 7.2, we get

$$Er(\mathcal{M}S)=A_1(\dot\kappa)+A_0(\dot{\mathbf{X}}_\alpha,\dot{\mathbf{X}}_\beta). \tag{8.22}$$

From (5.14), applying Lemma 7.1, Lemma 7.3, Lemma 7.4, Lemma 7.5, Lemma 7.6, we get

$$Er(\mathcal{M}T)=A_0(\dot\kappa,\dot{\mathbf{X}}_\alpha,\dot{\mathbf{X}}_\beta). \tag{8.23}$$

For the final piece, it is not necessary to consider the effect of the operator $\mathcal{M}$, so we dispense with it for convenience (be assured that the required result with $\mathcal{M}$ follows from the result without $\mathcal{M}$). From (5.23), applying Lemma 7.1, Lemma 7.2, we get

$$\begin{aligned}\dot K-\dot C_{\mathbf{n}}^h=&-\mathbf{K}_h^p[\mathbf{X}]\dot{\boldsymbol\eta}\cdot\mathbf{n}-\frac{1}{4\pi}\int_h^p\boldsymbol\eta\times Er\left(\hat{\mathbf{K}}\right)d\boldsymbol\alpha'\cdot\mathbf{n}-\frac{1}{4\pi}\int_h^p\mathbf{N}\times Er\left(\hat{\mathbf{K}}\right)d\boldsymbol\alpha'\cdot\frac{\boldsymbol\eta}{|\mathbf{N}|}\\&+A_0(\dot\kappa,\dot{\mathbf{X}}_\alpha,\dot{\mathbf{X}}_\beta)\\=&\frac{-1}{4\pi|\mathbf{N}|}\int_h^p\left(\mathbf{N}\cdot\left(\boldsymbol\eta'\times Er\left(\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|^3}\right)\right)+\boldsymbol\eta\cdot\left(\mathbf{N}'\times Er\left(\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|^3}\right)\right)\right)d\boldsymbol\alpha'\\&-\mathbf{K}_h^p[\mathbf{X}]\dot{\boldsymbol\eta}\cdot\mathbf{n}+A_0(\dot\kappa,\dot{\mathbf{X}}_\alpha,\dot{\mathbf{X}}_\beta)\end{aligned} \tag{8.24}$$

We note that on the right-hand side of (8.24), we have used the fact that

$$\frac{\dot{\boldsymbol{\eta}}}{|\mathbf{N}|}\int_h^p \mathbf{N}' \times \nabla_{\mathbf{x}} G(\mathbf{X}-\mathbf{X}')\, d\boldsymbol{\alpha}' = A_0(\dot{\kappa}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta); \tag{8.25}$$

this is true because on the left-hand side of (8.25), by the arguments of Section 6, this integral (which is identically zero in the continuous case) is, for instance, $O(h^3)$. This smallness allows for control of $\dot{\boldsymbol{\eta}}$ in the product. For the integral on the right-hand side of (8.24), we use the scalar triple product identity $\mathbf{A}_1 \cdot (\mathbf{A}_2 \times \mathbf{A}_3) = \mathbf{A}_3 \cdot (\mathbf{A}_1 \times \mathbf{A}_2)$; this yields the following:

$$\begin{aligned} \dot{K} - \dot{C}^h_{\mathbf{n}} = &- \frac{1}{4\pi |\mathbf{N}|}\int_h^p Er\left(\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|^3}\right) \cdot \big(\mathbf{N}\times \boldsymbol{\eta}' - \mathbf{N}' \times \boldsymbol{\eta}\big)\, d\boldsymbol{\alpha}' \\ &- \mathbf{K}^p_h[\mathbf{X}]\dot{\boldsymbol{\eta}} \cdot \mathbf{n} + A_0(\dot{\kappa}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta). \end{aligned} \tag{8.26}$$

Using Lemma 7.7 to gain one derivative, we get

$$\mathbf{K}^p_h[\mathbf{X}]\dot{\boldsymbol{\eta}} \cdot \mathbf{n} = A_0(\dot{\kappa}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta). \tag{8.27}$$

From its Taylor expansion, we can see that the term $\mathbf{N}\times\boldsymbol{\eta}' - \mathbf{N}'\times\boldsymbol{\eta}$ adds additional smoothing to the integral; we note that this smoothing in the integral is the reason to include the discretization of the term $C_{\mathbf{n}}$ in the numerical scheme. We can thus show that

$$\frac{1}{4\pi |\mathbf{N}|}\int_h^p Er\left(\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|^3}\right) \cdot \big(\mathbf{N}\times \boldsymbol{\eta}' - \mathbf{N}' \times \boldsymbol{\eta}\big)\, d\boldsymbol{\alpha}' = A_0(\dot{\mathbf{X}}). \tag{8.28}$$

Therefore, we have

$$\dot{K} - \dot{C}^h_{\mathbf{n}} = A_0(\dot{\kappa}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta, \dot{\mathbf{X}}). \tag{8.29}$$

Combining (8.21), (8.22), (8.23) and (8.29), we complete the proof. □

For the errors on the tangential velocities V_1, V_2, we have the following result.

Lemma 8.2. *The errors in the tangential velocities satisfy*

$$Er(\mathcal{M}V_1) = A_0(\dot{\kappa}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta, \dot{\mathbf{X}}), \tag{8.30}$$

$$Er(\mathcal{M}V_2) = A_0(\dot{\kappa}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta, \dot{\mathbf{X}}). \tag{8.31}$$

Proof. This is immediate from the definition (5.27), (5.28) and Lemma 8.1. □

In addition to Lemma 8.1 and Lemma 8.2, applying Lemma 7.1 multiple times, we obtain the evolution equations for the errors $\dot{\mathbf{X}}$, $\dot{\mathbf{X}}_\alpha$, $\dot{\mathbf{X}}_\beta$ and $\dot{\kappa}$. We state them in the following Lemma; these representations of these equations are fundamental to our ability to prove the energy estimate, and the introduction of the operator $\mathcal{M}$ in Section 5.2 is essential to our ability to give this lemma.

Lemma 8.3. *The evolution equations for the errors $\dot{\mathbf{X}}$, $\dot{\mathbf{X}}_\alpha$, $\dot{\mathbf{X}}_\beta$ and $\dot{\kappa}$ can be written as follows:*

$$\dot{\mathbf{X}}_t = A_1(\dot{\kappa}) + A_0(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^r), \tag{8.32}$$

$$\dot{\mathbf{X}}_{\alpha t} = A_2(\dot{\kappa}) + A_0(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^{r-1}), \tag{8.33}$$

$$\dot{\mathbf{X}}_{\beta t} = A_2(\dot{\kappa}) + A_0(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^{r-1}), \tag{8.34}$$

$$\dot{\kappa}_t = -\frac{B}{4E^{3/2}}\Lambda^3\dot{\kappa} + A_2(\dot{\kappa}) + A_{-3}(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^{r-2}). \tag{8.35}$$

Proof. These equations can be obtained from (5.37), (5.38), (5.39) and (5.43) respectively, using Lemma 8.1, Lemma 8.2, and Lemma 7.1, as well as the other lemmas such as Lemma 7.7. We note that the reason that on the right-hand side of 8.35, the form of the term $A_{-3}(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta))$ is due to the introduction of the $\mathcal{M}$ operator; without the operator $\mathcal{M}$, the corresponding terms would be much less regular. □

Recall that we have assumed (8.3) on $[0, T^*]$, for some sufficiently large constant $s > 0$. Now we use an energy estimate method to prove that the time interval on which (8.3) holds in fact can be extended to $[0, T]$ and this will complete the proof of Theorem 5.1. Let $(\cdot, \cdot)$ denote the inner product on L^2_h. In particular, for $f_h \in L^2_h$, we have $\|f_h\|^2_{L^2_h} = (f_h, f_h)$. The energy $\mathcal{E}$ is defined as follows:

$$\mathcal{E} = \frac{1}{2}(\Lambda^3\dot{\kappa}, \Lambda^3\dot{\kappa}) + \frac{1}{2}(\dot{\kappa}, \dot{\kappa}) + \frac{1}{2}(\dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\alpha) + \frac{1}{2}(\dot{\mathbf{X}}_\beta, \dot{\mathbf{X}}_\beta) + \frac{1}{2}(\dot{\mathbf{X}}, \dot{\mathbf{X}}). \tag{8.36}$$

This energy controls three derivatives of $\dot{\kappa}$, and zero derivatives of $\dot{\mathbf{X}}$, $\dot{\mathbf{X}}_\alpha$, and $\dot{\mathbf{X}}_\beta$. However, we note that we will be able to see the expected parabolic smoothing effect associated with Darcy flow, and we will thus be able to control more than three derivatives of $\dot{\kappa}$; this will be important in what follows.

To begin, we take the time derivative of (8.36):

$$\begin{aligned}\mathcal{E}_t =& \frac{1}{2}(\Lambda^3\dot{\kappa}, \Lambda^3\dot{\kappa})_t + \frac{1}{2}(\dot{\kappa}, \dot{\kappa})_t + \frac{1}{2}(\dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\alpha)_t + \frac{1}{2}(\dot{\mathbf{X}}_\beta, \dot{\mathbf{X}}_\beta)_t + \frac{1}{2}(\dot{\mathbf{X}}, \dot{\mathbf{X}})_t \\ =& (\Lambda^3\dot{\kappa}, \Lambda^3\dot{\kappa}_t) + (\dot{\kappa}, \dot{\kappa}_t) + (\dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_{\alpha t}) + (\dot{\mathbf{X}}_\beta, \dot{\mathbf{X}}_{\beta t}) + (\dot{\mathbf{X}}, \dot{\mathbf{X}}_t).\end{aligned} \tag{8.37}$$

Substituting from the evolution equations (8.32), (8.33), (8.34), and (8.35), this becomes

$$\begin{aligned}\mathcal{E}_t =& \left(\Lambda^3\dot{\kappa}, \Lambda^3\left(-\frac{B}{4E^{3/2}}\Lambda^3\dot{\kappa} + A_2(\dot{\kappa}) + A_{-3}(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^{r-2})\right)\right) \\ &+ (\dot{\kappa}, A_3(\dot{\kappa}) + A_{-3}(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^{r-2})) + (\dot{\mathbf{X}}_\alpha, A_2(\dot{\kappa}) + A_0(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^{r-1})) \\ &+ (\dot{\mathbf{X}}_\beta, A_2(\dot{\kappa}) + A_0(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^{r-1})) + (\dot{\mathbf{X}}, A_1(\dot{\kappa}) + A_0(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^r)).\end{aligned} \tag{8.38}$$

We continue to rewrite this; the important steps for now are that we write $(\Lambda^3\dot{\kappa}, \Lambda^3 A_2(\dot{\kappa})) = (A_4(\dot{\kappa}), A_4(\dot{\kappa}))$, and that we pull $\frac{1}{E^{3/2}}$ through $\Lambda^{3/2}$, incurring a commutator. Since the commutator of $\Lambda^{1/2}$ and a smooth function (such as $\frac{1}{E^{3/2}}$) is smoothing by one derivative (this can be seen by direct computation in Fourier space, but the interested reader could also consult equation (7.3) of [1]), the contribution of this commutator to the inner product is again of the form $(A_4(\dot{\kappa}), A_4(\dot{\kappa}))$. We thus have the following:

$$\begin{aligned}\mathcal{E}_t =& \left(-\frac{B}{4E^{3/2}}\Lambda^{9/2}\dot{\kappa}, \Lambda^{9/2}\dot{\kappa}\right) + (A_4(\dot{\kappa}), A_4(\dot{\kappa})) + \left(\Lambda^3\dot{\kappa}, A_0(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta)\right) + \left(\Lambda^3\dot{\kappa}, O(h^{r-5})\right) \\ &+ (\dot{\kappa}, A_3(\dot{\kappa}) + A_{-3}(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^{r-2})) + (\dot{\mathbf{X}}_\alpha, A_2(\dot{\kappa}) + A_0(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^{r-1})) \\ &+ (\dot{\mathbf{X}}_\beta, A_2(\dot{\kappa}) + A_0(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^{r-1})) + (\dot{\mathbf{X}}, A_1(\dot{\kappa}) + A_0(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta) + O(h^r)).\end{aligned} \tag{8.39}$$

Using elementary inequalities, we get the following:

$$\begin{aligned}\mathcal{E}_t \leq& -\frac{B}{4}(E^{-3/4}\Lambda^{9/2}\dot{\kappa}, E^{-3/4}\Lambda^{9/2}\dot{\kappa}) + (A_4(\dot{\kappa}), A_4(\dot{\kappa})) + C(\dot{\kappa}, \dot{\kappa}) + C(\Lambda^3\dot{\kappa}, \Lambda^3\dot{\kappa}) \\ &+ C(A_0(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta), A_0(\dot{\mathbf{X}}, \dot{\mathbf{X}}_\alpha, \dot{\mathbf{X}}_\beta)) + Ch^{r-5}\left\|\Lambda^3\dot{\kappa} + \dot{\kappa} + \dot{\mathbf{X}}_\alpha + \dot{\mathbf{X}}_\beta + \dot{\mathbf{X}}\right\|_{L^2_h}.\end{aligned} \tag{8.40}$$

Using Lemma 7.1, and using Young's Inequality, we can show that

$$(A_4(\dot{\kappa}), A_4(\dot{\kappa})) \leq \frac{B}{4}(E^{-3/4}\Lambda^{9/2}\dot{\kappa}, E^{-3/4}\Lambda^3\dot{\kappa}) + C_0\mathcal{E}, \tag{8.41}$$

for some constant $C_0 > 0$. We point out that this is true because the right-hand side includes terms controlling both more and fewer derivatives than are present on the left-hand side; to be specific, the right-hand side contains terms with both 9/2 spatial derivatives of $\dot{\kappa}$ and also zero derivatives of $\dot{\kappa}$. Furthermore, it is also necessary to know here that E is bounded, both from above and away from zero (cf. (2.23)). This term is then controlled, and we note that this is how we have used the beneficial parabolic smoothing.

Observing the important cancellation when combining (8.40) and (8.41), we arrive at the following:

$$\mathcal{E}_t \leq C'\mathcal{E}^{\frac{1}{2}}\left(\mathcal{E}^{\frac{1}{2}} + h^{r-5}\right), \tag{8.42}$$

for some other constant $C' > 0$. Then, using Gronwall's inequality, we get

$$\mathcal{E}(t) \leq C(T)h^{r-5} \tag{8.43}$$

where $C(T) > 0$ is a constant that does not depend on h, t or T^*. Finally, recall that r is chosen to be sufficiently large. In fact, as long as $r - 5 > s$, we can see that the assumption (8.3) holds on an interval greater than $[0, T^*]$. Hence we can extend $[0, T^*]$ until $T^* < T$. This completes the proof of Theorem 5.1.

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