Well-posedness of two-dimensional hydroelastic waves with mass

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Abstract

We study hydroelastic waves in interfacial flow of two-dimensional irrotational fluids. Each of the fluids is taken to be of infinite extent in one vertical direction, and bounded by a free surface in the other vertical direction. Elastic effects are considered at the free surface; this can describe physical settings such as the ocean bounded above by a layer of ice. A previous study proved well-posedness without considering the mass of the elastic surface; we now consider the effect of this mass. Under the assumption that a certain integral equation is solvable, we prove well-posedness of the initial value problem for the system. We are able to demonstrate that in some cases, such as the case of small mass parameter, the integral equation is indeed solvable. The proof uses geometric dependent variables, a normalized arclength parameterization, and a small-scale decomposition in the evolution equations.

1 Introduction

We study the initial value problem for an interfacial flow problem, in which two fluids whose velocities obey the incompressible, irrotational Euler equations are separated by a thin elastic sheet. This problem serves as a model for problems such as flapping flags [1] or ice sheets on the ocean [24]. We use the equations of motion for the problem as developed by Plotnikov and Toland [23].

One parameter in the equations of motion is the mass density along the elastic sheet. If this mass is neglected and the parameter set to zero, then a simpler set of equations results, still allowing for elastic effects. Most prior works in the literature for the hydroelastic problem have been concerned with the existence or computation of traveling waves, including by Toland in the massless case [27] and by Toland and Baldi and Toland in the case which accounts for the mass [9], [26]. Other papers on hydroelastic traveling waves include [16], [22], [28], [29].

We use a vortex sheet formulation of the problem; since the fluids are irrotational, there is no vorticity in the interior of either fluid region. However, the tangential velocities can jump across the elastic sheet, and thus there is measure-valued vorticity supported on the free surface. This measure-valued vorticity is proportional to the Dirac mass at the free surface, and this Dirac mass multiplies a function, γ, the vortex sheet strength. We must then evolve γ and the position of the free surface in order to have a solution of the problem.

Considering interfacial potential flows with surface tension accounted for at the free boundary, Hou, Lowengrub, and Shelley (HLS) introduced a non-stiff numerical method [18], [19]. The second author subsequently used the elements of their formulation of the problem to prove well-posedness for the vortex sheet with surface tension and for interfacial Darcy flow [3], [4]. The significant elements of the formulation involve making a geometric description of the free surface rather than using Cartesian coordinates, using an artificial tangential velocity to enforce a favorable parameterization, and isolating the leading-order terms in the evolution equation (i.e., making a small-scale decomposition). Other analytical works which subsequently used these ideas include [5], [6], [11], [12], [13], [14], [15], [31], [32].

Also using the HLS framework is the paper of the second author and Siegel which proves well-posedness of the hydroelastic initial value problem in the simpler case that the mass of the elastic sheet is neglected [7]. In the present, more general case, there are many additional terms to estimate in the evolution equation for γ. The greater difficulty, however, lies with the fact that the equation specifying the evolution of γ is actually an integral equation for γ_t; while this is also the case when the mass of the sheet is neglected, the form of the integral equation without mass is much more straightfoward. In particular, the integral equation which needed to be solved in [7] is the same integral equation which was proved to be solvable by Baker, Meiron, and Orszag in [8]. This result has been used many times in the literature, including, for example, in [3], [12], and [30]. In the present case, with a much more complicated integral equation to solve, we are not able to guarantee that it is always solvable. We prove well-posedness under the assumption that we may invert to find γ_t, with an estimate for the inverse operator which is analagous to the estimate developed in [12]. We are then able to demonstrate that this assumption holds in some cases, including in the case of small mass.

The plan of the paper is as follows: in Section 1.1, we introduce the norms and some operators which will be used in the sequel. In Section 2, we give a simple example which demonstrates how we will perform energy estimates for the hydroelastic problem with mass. In Section 3, we develop the equations of motion. In Section 4, we give some preliminary estimates, especially for operators such as certain commutators. Section 5 is devoted to our existence argument; this requires introducing a regularized system of equations, proving existence of solutions for the regularized system, proving an energy estimate which is uniform in the regularization parameter, and passing to the limit as this parameter vanishes. Section 6 proves uniqueness of solutions and continuous dependence upon the initial data.

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1.1 Function spaces, norms, operators, and notation

Derivatives with respect to the independent variables t and α will be denoted either by using the partial derivative operators ∂_t and ∂_α, or with subscripts. We comment now about the function spaces we will use. We use the L^2-based Sobolev spaces in the 2π-periodic setting. For $f \in H^k$ with $k \in \mathbb{N}$, we use the following as the norm:

$$\|f\|_k = \left(\int_0^{2\pi} f^2(\alpha) + (\partial_\alpha^k f(\alpha))^2 \, d\alpha \right)^{1/2}.$$

For$f \in H^{k+1/2}$, with $k \in \mathbb{N}$, we use the following as the norm:

$$\|f\|_{k+1/2} = \left(\int_0^{2\pi} f^2(\alpha) + \left((\partial_\alpha^k f(\alpha)) H \partial_\alpha^{k+1} f(\alpha) \right) \, d\alpha \right)^{1/2}.$$

Here, H is the periodic Hilbert transform, which has symbol $\hat{H}(\xi) = -i\mathrm{sgn}(\xi)$. (Notice that if f has mean zero, then $H^2 f = -f$. For more information on the periodic Hilbert transform, the interested reader might consult [17].) Using Plancherel's Theorem, it is clear that

$$\|f\|_{k+1/2} = \left(\sum_\xi (1 + |\xi|^{2k+1}) |\hat{f}(\xi)|^2 \right)^{1/2},$$

so this is equivalent to any other usual definition of the $H^{k+1/2}$ norm. We introduce the operator $\Lambda = H\partial_\alpha$, which will be useful many times in what follows; note that the symbol of Λ is $\hat{\Lambda}(\xi) = |\xi|$, and this implies that Λ is self-adjoint. This implies the following, which we will use many times:

$$\frac{d}{dt} \int_0^{2\pi} g \Lambda g d\alpha = 2 \int_0^{2\pi} g \Lambda g_t d\alpha = 2 \int_0^{2\pi} g_t \Lambda g \, d\alpha. \quad (1)$$

This will be relevant as we estimate the growth of quantities which are equivalent to $H^{k+1/2}$ norms. We will sometimes use the projection which removes the zero mode of a periodic function:

$$\mathbb{P}f = f - \frac{1}{2\pi} \int_0^{2\pi} f(\alpha) d\alpha.$$

We may sometimes denote the mean of a periodic function as $\langle\langle f \rangle\rangle$, so that we could say $\mathbb{P}f = f - \langle\langle f \rangle\rangle$. We also introduce the mean-zero antiderivative operator, ∂_α^{-1}. This is defined through its symbol as $\widehat{\partial_\alpha^{-1} f}(\xi) = \frac{1}{i\xi} \hat{f}(\xi)$, for $\xi \neq 0$, and $\widehat{\partial_\alpha^{-1} f}(0) = 0$. Notice that if f has zero mean, then $f = \partial_\alpha \partial_\alpha^{-1} f = \partial_\alpha^{-1} \partial_\alpha f$.

Remark 1. In the existence theorem that follows, we will be proving existence for $(\theta, \gamma) \in H^s \times H^{s-1}$; naturally, θ and γ will defined soon. This remark, though, concerns the value of s. We will say several times that s is taken to be "sufficiently large." This means that there exists $s_0 \in \mathbb{Z}$ such that if $s \geq s_0$, then all of our arguments work. We have not counted exactly the value of this s_0, but we do note in a few places in the sequel that we take $s \geq 5$. A careful reading of all of the arguments would allow one to compute exactly this minimal value of s_0. We further note that this approach to the regularity is the same as taken by the second author in papers such as [3] and [5].

2 An instructive example

Let both c_1 and c_2 be positive constants. We consider the following linear system, which has the same types of leading-order terms as the hydroelastic system with mass:

$$\theta_t = H(\gamma_\alpha), \tag{2}$$

$$\gamma_t = -c_1 H(\gamma_{\alpha t}) - c_2\theta_{\alpha\alpha\alpha\alpha} + c_3\theta_{\alpha\alpha} + c_4 H(\gamma_{\alpha\alpha}) + c_5 H(\gamma_\alpha). \tag{3}$$

While we assume that c_1 and c_2 are positive, we are able to estimate the contributions from the remaining terms without regard to the signs of c_3, c_4, and c_5. For the purpose of the present example, we take (θ, γ) to be a solution of this system which is sufficiently smooth for all of the integrals we are about to use to make sense.

It is obvious that the operator $(I + c_1\Lambda)^{-1}$ exists and is a bounded operator from $H^s \to H^{s+1}$. So we can rewrite (3) as follows:

$$\gamma_t = (I + c_1\Lambda)^{-1}(-c_2\theta_{\alpha\alpha\alpha\alpha} + c_3\theta_{\alpha\alpha} + c_4 H(\gamma_{\alpha\alpha}) + c_5 H(\gamma_\alpha)). \tag{4}$$

The energy functional we will estimate will serve as an upper bound for a constant times the square of the H^3-norm of θ and the square of the H^2-norm of γ. We let $E(t)$ be given by

$$E(t) = E_0(t) + E_1(t) + E_2(t) + E_3(t),$$

where

$$E_0(t) = \frac{1}{2}\int_0^{2\pi} \theta^2 + \gamma^2 \, d\alpha,$$
$$E_1(t) = \frac{c_2}{2}\int_0^{2\pi} (\partial_\alpha^3\theta)^2 \, d\alpha,$$
$$E_2(t) = \frac{1}{2}\int_0^{2\pi} (\partial_\alpha\gamma)(H\partial_\alpha^2\gamma) \, d\alpha,$$
$$E_3(t) = \frac{c_1}{2}\int_0^{2\pi} (\partial_\alpha^2\gamma)(\partial_\alpha^2\gamma) \, d\alpha.$$

Taking the time derivative and substituting from (2) and (4), we have

$$\frac{dE_0}{dt} = \int_0^{2\pi} \theta H\gamma_\alpha + \gamma(I + c_1\Lambda)^{-1}(-c_2\theta_{\alpha\alpha\alpha\alpha} + c_3\theta_{\alpha\alpha} + c_4 H(\gamma_{\alpha\alpha}) + c_5 H(\gamma_\alpha)) \, d\alpha \leq CE, \tag{5}$$

since the integral can be bounded in terms of at most third derivatives of θ and at most the first derivative of γ because of the mapping property of $(I + c_1\Lambda)^{-1}$.

We next take the time derivative of E_1 and substitute from (2):

$$\frac{dE_1}{dt} = c_2\int_0^{2\pi} (\partial_\alpha^3\theta)(\partial_\alpha^3\theta_t) d\alpha = c_2\int_0^{2\pi} (\partial_\alpha^3\theta)(H\partial_\alpha^4\gamma) \, d\alpha. \tag{6}$$

Next, we take the time derivative of E_2 and plug in from (3):

$$\begin{aligned}\frac{dE_2}{dt} &= \int_0^{2\pi} (\partial_\alpha \gamma_t)(H\partial_\alpha^2\gamma)\, d\alpha \\ &= -c_1 \int_0^{2\pi} (H\partial_\alpha^2\gamma_t)(H\partial_\alpha^2\gamma)\, d\alpha - c_2 \int_0^{2\pi} (\partial_\alpha^5\theta)(H\partial_\alpha^2\gamma)\, d\alpha + c_3 \int_0^{2\pi} (\partial_\alpha^3\theta)(H\partial_\alpha^2\gamma)\, d\alpha \\ &+ c_4 \int_0^{2\pi} H(\partial_\alpha^3\gamma)(H\partial_\alpha^2\gamma)\, d\alpha + c_5 \int_0^{2\pi} H(\partial_\alpha^2\gamma)(H\partial_\alpha^2\gamma)\, d\alpha.\end{aligned}$$

We rewrite the first and fourth terms on the right-hand side; for the first term, we use the fact that the adjoint of H is $-H$, and for the fourth term, we recognize a perfect derivative:

$$\begin{aligned}\frac{dE_2}{dt} =& - c_1 \int_0^{2\pi} (\partial_\alpha^2\gamma_t)(\partial_\alpha^2\gamma)\, d\alpha - c_2 \int_0^{2\pi} (\partial_\alpha^5\theta)(H\partial_\alpha^2\gamma)\, d\alpha + c_3 \int_0^{2\pi} (\partial_\alpha^3\theta)(H\partial_\alpha^2\gamma)\, d\alpha \\ &+ \frac{c_4}{2} \int_0^{2\pi} \partial_\alpha((H\partial_\alpha^2\gamma))^2\, d\alpha + c_5 \int_0^{2\pi} H(\partial_\alpha^2\gamma)(H\partial_\alpha^2\gamma)\, d\alpha.\end{aligned}$$

Recognizing that the first term on the right-hand side is equal to $-dE_3/dt$, and that the fourth term on the right-hand side integrates to zero, this becomes

$$\frac{dE_2}{dt} = -\frac{dE_3}{dt} - c_2 \int_0^{2\pi} (\partial_\alpha^5\theta)(H\partial_\alpha^2\gamma)\, d\alpha + c_3 \int_0^{2\pi} (\partial_\alpha^3\theta)(H\partial_\alpha^2\gamma)\, d\alpha + c_5 \int_0^{2\pi} H(\partial_\alpha^2\gamma)(H\partial_\alpha^2\gamma)\, d\alpha. \quad (7)$$

We now add (6) and (7). An important cancellation occurs upon integrating by parts, and we slightly rearrange the remaining terms, finding the following:

$$\frac{dE_1}{dt} + \frac{dE_2}{dt} + \frac{dE_3}{dt} = c_3 \int_0^{2\pi} (\partial_\alpha^3\theta)(H\partial_\alpha^2\gamma)\, d\alpha + c_5 \int_0^{2\pi} H(\partial_\alpha^2\gamma)(H\partial_\alpha^2\gamma) d\alpha \le CE. \quad (8)$$

Adding (5) and (8), we have

$$\frac{dE}{dt} \le CE \quad (9)$$

This implies that the energy grows at most exponentially. Recall that c_1 and c_2 are positive; we have the following conclusion:

$$\min\{\frac{1}{2}, \frac{c_1}{2}, \frac{c_2}{2}\}(\|\theta\|_3^2 + \|\gamma\|_2^2) \le E(t) \le E(0)e^{Ct}. \quad (10)$$

3 Equations of motion

The model we consider is an elastic sheet with density ρ_0 between two irrotational, inviscid, incompressible fluids. The densities of the lower and upper fluids are denoted by ρ_1, ρ_2, respectively. The one-dimensional free surface is $\boldsymbol{X}(\alpha, t) = (x(\alpha, t), y(\alpha, t))$ with period 2π, where α is the parameter along the curve and t is time; specifically, the periodicity means that $(x(\alpha, t), y(\alpha, t))$ satisfies

$$x(\alpha + 2\pi, t) = x(\alpha, t) + 2\pi, \quad y(\alpha + 2\pi, t) = y(\alpha, t), \qquad \forall \alpha, t. \quad (11)$$

We let $\boldsymbol{t}, \boldsymbol{n}$ denote the unit tangent and normal vectors along the curve, and s is arclength. These quantities $\boldsymbol{t}$, $\boldsymbol{n}$, and s_α are defined by

$$\begin{aligned}
&\boldsymbol{t} = \frac{(x_\alpha, y_\alpha)}{s_\alpha}, \quad \boldsymbol{n} = \frac{(-y_\alpha, x_\alpha)}{s_\alpha}, \\
&s_\alpha^2 = x_\alpha^2 + y_\alpha^2.
\end{aligned}$$

We denote U and V as the normal and tangential velocities of the free surface, so that

$$\boldsymbol{X}_t = U\boldsymbol{n} + V\boldsymbol{t}. \tag{12}$$

We define the tangent angle $\theta = \tan^{-1}(y_\alpha/x_\alpha)$. Then the evolution equations for s_a and θ, which we infer from (12), are

$$\begin{aligned}
s_{\alpha t} &= V_\alpha - \theta_\alpha U, \\
\theta_t &= \frac{U_\alpha + V\theta_\alpha}{s_\alpha}.
\end{aligned} \tag{13}$$

We introduce κ, the curvature of the curve $\boldsymbol{X}$. The relationship between curvature and tangent angle is

$$\kappa = \theta_\alpha / s_\alpha.$$

Notice that the normal velocity U is determined by the physics of the problem; however, the tangential velocity V is not. That is to say, we have some freedom when choosing V, and we use this freedom to enforce our preferred parameterization. Our preferred parameterization is a normalized arclength parameterization: we would like s_α to be independent of the variable α. If we denote by $L(t)$ the length of one period of the curve, then we would like

$$s_\alpha = L/2\pi, \tag{14}$$

for all t. If the equation (14) holds at the initial time, then it will hold at later time as long as

$$s_{\alpha t} = L_t/2\pi. \tag{15}$$

Then we can solve for V by the equation (13) and equation (15), as long as we know L_t. To this end, we calculate that $L_t = 2\pi s_{\alpha t}$, and we average this over the interval $[0, 2\pi]$:

$$L_t = \int_0^{2\pi} s_{\alpha t} d\alpha = -\int_0^{2\pi} \theta_\alpha U \, d\alpha. \tag{16}$$

Here, we have used the fact that V_α integrates to zero over $[0, 2\pi]$, since V is periodic. Using the equation (13) and equation (16), we have

$$V_\alpha = L_t/2\pi + \theta_\alpha U = -\frac{1}{2\pi}\int_0^{2\pi} \theta_\alpha U d\alpha + \theta_\alpha U = \mathbb{P}(\theta_\alpha U).$$

Therefore the equation of V is

$$V = \partial_\alpha^{-1}\mathbb{P}(\theta_\alpha U) + V_0(t). \tag{17}$$

We will comment on the choice of $V_0(t)$, which is the mean of V at each time, at the end of Section 3.1.

Since there is no vorticity in the bulk of the fluid, we are able to use a vortex sheet formulation. The average of the upper and lower fluid velocities evaluated at the interface is denoted by $\boldsymbol{W} = (W_1, W_2)$ and is specified by the Birkhoff-Rott integral. The Birkhoff-Rott integral and its approximation will be introduced in Section 3.1. The normal velocity is the normal component of the Birkhoff-Rott integral,

$$U = \boldsymbol{W} \cdot \boldsymbol{n}. \tag{18}$$

In the paper, we denote

$$V_W = V - \boldsymbol{W} \cdot \boldsymbol{t}. \tag{19}$$

We use the new notation to rewrite θ_t :

$$\theta_t = \frac{1}{s_\alpha}(\boldsymbol{W}_\alpha \cdot \boldsymbol{n} - V_W \theta_\alpha) \tag{20}$$

Establishing (20) requires use of the geometric identity $\boldsymbol{n}_\alpha = -\theta_\alpha \boldsymbol{t}$.

The vortex sheet strength γ is the jump in velocity across the interface. Since the velocity potential on each side of the interface satisfies a Bernoulli equation, upon taking the limit at the interface, an evolution equation for the jump in potential across the interface can be found. Differentiating this leads to the following evolution equation for γ:

$$\gamma_t = -\frac{2}{\rho_1 + \rho_2}[P]_\alpha + \frac{(V_W\gamma)_\alpha}{s_\alpha} - 2A\left(s_\alpha \boldsymbol{W}_t \cdot \boldsymbol{t} + \frac{1}{8}\partial_\alpha\left(\frac{\gamma^2}{s_\alpha^2}\right) - V_W \boldsymbol{W}_\alpha \cdot \boldsymbol{t} + g y_\alpha\right), \tag{21}$$

where $[P] = p_1 - p_2$ is the jump in pressure at the interface, g is the acceleration due to gravity, and $A = \dfrac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$. The equation (21) is given in [7]; the work [7] treated the case with zero density of the elastic sheet, $\rho_0 = 0$. At present we instead consider $\rho_0 > 0$. The model of Plotnikov and Toland [23] gives the equation of motion of the elastic sheet as

$$\rho\partial_{tt}\boldsymbol{X} + \big\{\frac{1}{2}\Delta W_b'(H) + 2H(HW_b'(H) - W_b(H) - KW_b'(H))\big\}\boldsymbol{n} = P(\boldsymbol{X})\boldsymbol{n} - g\rho\boldsymbol{j}, \tag{22}$$

where $\boldsymbol{j} = (0, 1)$, H is the mean curvature, K is the Gauss curvature, and $\rho = \rho_0/\sqrt{a}$ is the density of the deformed elastic shell at point $\boldsymbol{X}$, with $\sqrt{a}$ related to the metric of the free surface. In the sequel, we take $W_b(H)$ to be a quadratic function of b, namely $W_b(H) = E_b H^2$, where the constant E_b is the bending modulus.

In two dimensions, $\boldsymbol{X}(\alpha, t) = (x(\alpha, t), y(\alpha, t))$, $\Delta = \partial_{ss}$, $K = 0$, and $H = \kappa/2$, where s, κ are the arclength and the curvature of the curve, respectively. The first-order geometric invariant $\sqrt{a}$ is given by $\sqrt{a} = \sqrt{x_\alpha^2 + y_\alpha^2} = s_\alpha$. Then, the equation (22) becomes

$$\rho\partial_{tt}\boldsymbol{X} + \frac{1}{2}E_b\big\{\kappa_{ss} + \frac{1}{2}\kappa^3\big\}\boldsymbol{n} = P(\boldsymbol{X})\boldsymbol{n} - g\rho\boldsymbol{j}. \tag{23}$$

The equation for the jump in pressure across the interface, $P(\boldsymbol{X})$, is then given by

$$P(\boldsymbol{X}) = \rho\partial_{tt}\boldsymbol{X}\cdot\boldsymbol{n} + \frac{1}{2}E_b\{\kappa_{ss} + \frac{1}{2}\kappa^3\} + g\rho\boldsymbol{j}\cdot\boldsymbol{n}. \tag{24}$$

By (12), we calculate $\partial_{tt}\boldsymbol{X}\cdot\boldsymbol{n}$:

$$\begin{aligned}\partial_{tt}\boldsymbol{X}\cdot\boldsymbol{n} &= (U_t\boldsymbol{n} + V_t\boldsymbol{t} - U\boldsymbol{t}\theta_t + V\boldsymbol{n}\theta_t)\cdot\boldsymbol{n} = U_t + V\theta_t \\ &= \boldsymbol{W}_t\cdot\boldsymbol{n} - \boldsymbol{W}\cdot\boldsymbol{t}\theta_t + V\theta_t = \boldsymbol{W}_t\cdot\boldsymbol{n} + V_W\theta_t.\end{aligned}$$

Here, we have used the geometric identities $\boldsymbol{t}_t = \theta_t\boldsymbol{n}$ and $\boldsymbol{n}_t = -\theta_t\boldsymbol{t}$, as well as (18). We need to calculate a derivative of this with respect to α :

$$\big(\partial_{tt}\boldsymbol{X}\cdot\boldsymbol{n}\big)_\alpha = \boldsymbol{W}_{\alpha t}\cdot\boldsymbol{n} - \boldsymbol{W}_t\cdot\boldsymbol{t}\theta_\alpha + (V_W)_\alpha\theta_t + V_W\theta_{t\alpha}, \tag{25}$$

where we have again used the geometric identity $\boldsymbol{n}_\alpha = -\theta_\alpha\boldsymbol{t}$. Substituting the equation for the derivative of the jump in pressure, $[P]_\alpha$, we rewrite γ_t as

$$\begin{aligned}\gamma_t = &-\frac{1}{\rho_1+\rho_2}E_b\big(\kappa_{ss} + \frac{1}{2}\kappa^3\big)_\alpha + \frac{(V_W\gamma)_\alpha}{s_\alpha} - 2A\left(s_\alpha\boldsymbol{W}_t\cdot\boldsymbol{t} + \frac{1}{8}\partial_\alpha\left(\frac{\gamma^2}{s_\alpha^2}\right) - V_W\boldsymbol{W}_\alpha\cdot\boldsymbol{t} + gy_\alpha\right) \\ &- \frac{2\rho}{\rho_1+\rho_2}\big(\boldsymbol{W}_{\alpha t}\cdot\boldsymbol{n} - \boldsymbol{W}_t\cdot\boldsymbol{t}\theta_\alpha + (V_W)_\alpha\theta_t + V_W\theta_{t\alpha} + g(x_\alpha/s_\alpha)_\alpha\big).\end{aligned}$$

Using $\kappa = \theta_\alpha/s_\alpha$, $\partial_s = \partial_\alpha/s_\alpha$, and $s_\alpha = \dfrac{L}{2\pi}$, and defining $\widetilde{A} = \dfrac{\rho}{\rho_1+\rho_2}$ and $\bar{A} = \dfrac{8\pi^3E_b}{L^3(\rho_1+\rho_2)}$, we may restate the evolution of γ as follows:

$$\begin{aligned}\gamma_t = &-\bar{A}\left(\partial_\alpha^4\theta + \frac{3\theta_\alpha^2\theta_{\alpha\alpha}}{2}\right) + \frac{2\pi(V_W\gamma)_\alpha}{L} - 2\widetilde{A}(\boldsymbol{W}_{\alpha t}\cdot\boldsymbol{n}) - \big(2A - 4\pi\widetilde{A}\theta_\alpha/L\big)(\boldsymbol{W}_t\cdot\boldsymbol{t})s_\alpha \\ &-2\widetilde{A}\left((V_W)_\alpha\theta_t + V_W\theta_{t\alpha} + \frac{2\pi gx_{\alpha\alpha}}{L}\right) - 2A\left(\left(\frac{\pi^2}{L^2}\right)\gamma\gamma_\alpha - V_W\boldsymbol{W}_\alpha\cdot\boldsymbol{t} + gy_\alpha\right).\end{aligned} \tag{26}$$

We have mentioned before that we will focus on the elastic sheet with mass between two fluids. Comparing to the elastic sheet between two fluids without accounting for its mass, it is obvious that the evolution of θ_t is the same. However, the evolution of γ_t is significantly more complicated in the presence of mass, i.e., in the case $\widetilde{A} > 0$. The additional terms which arise in the γ_t equation, such as $\boldsymbol{W}_{\alpha t}\cdot\boldsymbol{n}$, $(V_W)_\alpha\theta_t$ and $V_W\theta_{t\alpha}$, require care to understand and estimate.

3.1 The Birkhoff-Rott integral, related operators, and consequences

To better understand the Birkhoff-Rott integral, it is helpful to introduce a bit of complex notation. We define a complexification map: $\mathcal{C}:\mathbb{R}^2\to\mathbb{C}$ to be

$$\mathcal{C}(x,y) = x + iy. \tag{27}$$

We denote a complex field point $z = x + iy = \mathcal{C}(x, y)$. The Birkhoff-Rott integral is

$$\mathcal{C}(\boldsymbol{W})^* = W_1 - iW_2 = \frac{1}{4\pi i} PV \int_0^{2\pi} \gamma(\alpha') \cot(\frac{1}{2}(z(\alpha) - z(\alpha')))\, d\alpha'. \tag{28}$$

Applying the complexification map, we have the following:

$$\mathcal{C}(\boldsymbol{t}) = z_\alpha / s_\alpha, \tag{29}$$
$$\mathcal{C}(\boldsymbol{n}) = iz_\alpha / s_\alpha, \tag{30}$$
$$\boldsymbol{W} \cdot \boldsymbol{n} = Re\{\mathcal{C}(\boldsymbol{W})^* \mathcal{C}(\boldsymbol{n})\}. \tag{31}$$

We introduced the periodic Hilbert transform, H, in the introduction. However, we only discussed its symbol. There is also an integral form of the periodic Hilbert transform; if $f \in L^2$, then we have

$$Hf(\alpha) = \frac{1}{2\pi} PV \int_0^{2\pi} f(\alpha') \cot(\frac{1}{2}(\alpha - \alpha'))\, d\alpha'. \tag{32}$$

Notice that, the Birkhoff-Rott integral (28) looks very much like the Hilbert transform. Thus, it is natural to introduce the following operator, which is the error from approximating the Birkhoff-Rott integral using the Hilbert transform. In Section 4, we will see that the error operator is a smooth operator as desired. The error operator is defined as follows:

$$K[z_d]f(\alpha) = \frac{1}{4\pi i} \int_0^{2\pi} f(\alpha') \left[\cot(\frac{1}{2}(z_d(\alpha) - z_d(\alpha'))) - \frac{1}{z_\alpha(\alpha')} \cot(\frac{1}{2}(\alpha - \alpha'))\right] d\alpha'. \tag{33}$$

Here, we have introduced the quantity z_d, which is defined as

$$z_d(\alpha, t) = z(\alpha, t) - z(0, t);$$

it is necessary to introduce z_d, since z_d is determined uniquely from θ while z is not. Notice that $z_d(\alpha, t) - z_d(\alpha', t) = z(\alpha, t) - z(\alpha', t)$ and $\partial_\alpha z_d = \partial_\alpha z$.

We will also need to introduce the commutator of the Hilbert transform and multiplication by a smooth function ϕ, which is

$$[H, \phi]f(\alpha) = H(\phi f)(\alpha) - \phi(\alpha) H(f)(\alpha).$$

We will prove that the operator $[H, \phi]$ is also a smoothing operator in Section 4.

To rewrite the θ_t equation, we need a formula for $\boldsymbol{W}_\alpha$, and we will use the operators we have introduced for this purpose. After some manipulations, we arrive at the following (see [3] for the full details):

$$\begin{aligned}\mathcal{C}(\boldsymbol{W})^*_\alpha &= \frac{1}{4\pi i} PV \int_0^{2\pi} z_\alpha(\alpha) \partial_{\alpha'} \left(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}\right) \cot(\frac{1}{2}(z(\alpha) - z(\alpha')))\, d\alpha' \\ &= z_\alpha K[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) + \frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha \left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) + \frac{1}{2iz_\alpha} H\left(\gamma_\alpha - \gamma \frac{z_{\alpha\alpha}}{z_\alpha}\right).\end{aligned}$$

Let $\boldsymbol{m}$ denote the quantity defined by

$$\mathcal{C}(\boldsymbol{m})^* = z_\alpha K[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) + \frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right). \tag{34}$$

Then, we arrive at a useful formula for $\boldsymbol{W}_\alpha$:

$$\boldsymbol{W}_\alpha = \frac{\pi}{L}H(\gamma_\alpha)\boldsymbol{n} - \frac{\pi}{L}H(\gamma\theta_\alpha)\boldsymbol{t} + \boldsymbol{m}. \tag{35}$$

Again, the interested reader might see [3] for full details; the formulas for $\boldsymbol{W}_\alpha$ and $\boldsymbol{m}$ were initially developed by the second author in [3] and were used subsequently in several papers, including [4], [5], [6], and [7]. We substitute (35) into (20), finding the following useful form of the the θ_t equation:

$$\theta_t = \frac{2\pi^2}{L^2}H(\gamma_\alpha) + \frac{2\pi}{L}V_W\theta_\alpha + \frac{2\pi}{L}\boldsymbol{m}\cdot\boldsymbol{n}. \tag{36}$$

We use (35) to give a useful formula for V_W, which was defined previously in (19). To begin, we differentiate V_W :

$$\partial_\alpha V_W = V_\alpha - \boldsymbol{W}_\alpha\cdot\boldsymbol{t} - (\boldsymbol{W}\cdot\boldsymbol{n})\theta_\alpha.$$

Since $V_\alpha = \dfrac{L_t}{2\pi} + \theta_\alpha U$ and $U = \boldsymbol{W}\cdot\boldsymbol{n}$, we see that

$$\partial_\alpha V_W = \frac{\pi}{L}H(\gamma\theta_\alpha) - \boldsymbol{m}\cdot\boldsymbol{t} + \frac{L_t}{2\pi}.$$

This can be rewritten by using the operator $\mathbb{P}$; recall that $\mathbb{P}$ removes the mean of a periodic function. Since the left-hand side has zero mean, and since a Hilbert transform has zero mean, we must have the following:

$$\partial_\alpha V_W = \frac{\pi}{L}H(\gamma\theta_\alpha) - \mathbb{P}(\boldsymbol{m}\cdot\boldsymbol{t}).$$

We apply the operator ∂_α^{-1}, finding

$$V_W = \partial_\alpha^{-1}\left(\frac{\pi}{L}H(\gamma\theta_\alpha) - \mathbb{P}(\boldsymbol{m}\cdot\boldsymbol{t})\right). \tag{37}$$

From the formula of V_W, we can see that the mean of V is same as the mean of $\boldsymbol{W}\cdot\boldsymbol{t}$. This is possible since the equation of V is defined from V_α, and the mean of V is then free to be chosen. Previously, we denoted the mean of V at each time as $V_0(t)$, so we see that we have taken this to be the mean of $\boldsymbol{W}\cdot\boldsymbol{t}$ at each time.

3.2 Calculation of $(\boldsymbol{W}_t\cdot\boldsymbol{t})s_\alpha$ and $\boldsymbol{W}_{\alpha t}\cdot\boldsymbol{n}$

To analyze the right-hand side of the equation for γ_t, we must find good expressions for $(\boldsymbol{W}_t\cdot\boldsymbol{t})s_\alpha$ and $\boldsymbol{W}_{\alpha t}\cdot\boldsymbol{n}$.

To begin, we can write $\boldsymbol{W}_t$ as

$$\begin{aligned}\mathcal{C}(\boldsymbol{W}_t)^* = \frac{1}{4\pi i} PV \int_0^{2\pi} \gamma_t(\alpha')\cot(\frac{1}{2}(z(\alpha) - z(\alpha')))\, d\alpha' \\ -\frac{1}{8\pi i} PV \int_0^{2\pi} \gamma(\alpha')(z_t(\alpha) - z_t(\alpha'))\csc^2(\frac{1}{2}(z(\alpha) - z(\alpha')))\, d\alpha'.\end{aligned} \tag{38}$$

Then we can write $(\boldsymbol{W}_t \cdot \boldsymbol{t})s_\alpha$ as

$$\begin{aligned}(\boldsymbol{W}_t \cdot \boldsymbol{t})s_\alpha &= Re\{\mathcal{C}(\boldsymbol{W}_t)^* z_\alpha\} \\ &= Re\left\{\frac{z_\alpha}{4\pi i} PV \int_0^{2\pi} \gamma_t(\alpha')\cot(\frac{1}{2}(z(\alpha) - z(\alpha')))\, d\alpha'\right\} \\ &- Re\left\{\frac{z_\alpha}{8\pi i} PV \int_0^{2\pi} \gamma(\alpha')(z_t(\alpha) - z_t(\alpha'))\csc^2(\frac{1}{2}(z(\alpha) - z(\alpha')))\, d\alpha'\right\}.\end{aligned}$$

We define the first term on the right-hand side to be $J[z_d]\gamma_t$ and the second term to be R_0.

Using integration by parts, R_0 can be rewritten as the sum of R_1 and R_2, where

$$\begin{aligned}R_1 &= Re\left\{\frac{z_\alpha}{4\pi i} PV \int_0^{2\pi} \left(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}\right)_{\alpha'} (z_t(\alpha) - z_t(\alpha'))\cot(\frac{1}{2}(z(\alpha) - z(\alpha')))\, d\alpha'\right\}, \\ R_2 &= -Re\left\{\frac{z_\alpha}{4\pi i} PV \int_0^{2\pi} \frac{\gamma(\alpha')z_{t\alpha}(\alpha')}{z_\alpha(\alpha')}\cot(\frac{1}{2}(z(\alpha) - z(\alpha')))\, d\alpha'\right\}.\end{aligned}$$

Of course, we can continue to rewrite R_1 and R_2. Fortunately, the terms comprising R_1 will be unimportant in the sequel:

$$R_1 = Re\left\{z_\alpha z_t K[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)\right\} - Re\left\{z_\alpha K[z_d]\left(z_t\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)\right\} - Re\left\{\frac{z_\alpha}{2i}[H, z_t]\left(\frac{1}{z_\alpha}\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)\right\}. \tag{39}$$

By adding and subtracting in R_2, we have

$$\begin{aligned}R_2 &= -Re\left\{z_\alpha K[z_d]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\} - Re\left\{\frac{z_\alpha}{2i} H\left(\frac{\gamma z_{t\alpha}}{z_\alpha^2}\right)\right\} \\ &= -Re\left\{z_\alpha K[z_d]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\} - Re\left\{\frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha}\right]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\} - Re\left\{\frac{1}{2i} H\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\}.\end{aligned}$$

The last term on the right-hand side of this equation can be simplified because of the following calculation:

$$Re\left\{\frac{z_{t\alpha}}{iz_\alpha}\right\} = \frac{(x_{\alpha t}, y_{\alpha t}) \cdot \boldsymbol{n}}{s_\alpha} = \frac{U_\alpha + V\theta_\alpha}{s_\alpha} = \theta_t.$$

Using this fact, we find the following formula for R_2 :

$$R_2 = -Re\left\{z_\alpha K[z_d]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\} - Re\left\{\frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha}\right]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\} - \frac{1}{2} H\left(\gamma\theta_t\right). \tag{40}$$

The first two terms on the right-hand side of (40) involve the operator $K[z_d]$ and commutator operators. Good estimates are available for both of these (we will present these estimates in Section 4 below). We will deal with the last term on the right-hand side of (40) carefully later.

To calculate $\boldsymbol{W}_{\alpha t}\cdot\boldsymbol{n}$, we will introduce $\mathcal{C}(\boldsymbol{W})^*_{\alpha t}$ first:

$$\begin{aligned}
\mathcal{C}(\boldsymbol{W})^*_{\alpha t}(\alpha) &= \partial_t\bigg(\frac{1}{4\pi i}PV\int_0^{2\pi} z_\alpha(\alpha)\bigg(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}\bigg)_{\alpha'}\cot\big(\frac{1}{2}(z(\alpha)-z(\alpha'))\big)\,d\alpha'\bigg)\\
&= \frac{z_{\alpha t}(\alpha)}{4\pi i}PV\int_0^{2\pi}\bigg(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}\bigg)_{\alpha'}\cot\big(\frac{1}{2}(z(\alpha)-z(\alpha'))\big)\,d\alpha'\\
&+\frac{z_{\alpha}(\alpha)}{4\pi i}PV\int_0^{2\pi}\bigg(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}\bigg)_{\alpha' t}\cot\big(\frac{1}{2}(z(\alpha)-z(\alpha'))\big)\,d\alpha'\\
&-\frac{z_{\alpha}(\alpha)}{8\pi i}PV\int_0^{2\pi}\bigg(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}\bigg)_{\alpha'}(z_t(\alpha)-z_t(\alpha'))\csc^2\big(\frac{1}{2}(z(\alpha)-z(\alpha'))\big)\,d\alpha'\\
&=: \mathcal{C}(\boldsymbol{A}_1)^*+\mathcal{C}(\boldsymbol{A}_2)^*+\mathcal{C}(\boldsymbol{A}_3)^*.
\end{aligned}$$

So $\boldsymbol{W}_{\alpha t}\cdot\boldsymbol{n}=\boldsymbol{A_1}\cdot\boldsymbol{n}+\boldsymbol{A_2}\cdot\boldsymbol{n}+\boldsymbol{A_3}\cdot\boldsymbol{n}$. We will investigate these terms separately.

In what follows, we will use $z_\alpha = s_\alpha e^{i\theta}$. We begin with $\boldsymbol{A}_1$. Recalling the definition of $\boldsymbol{m}$, we may write $\boldsymbol{A}_1$ as follows:

$$\mathcal{C}(\boldsymbol{A}_1)^* = \frac{z_{\alpha t}}{2iz_\alpha^2}H(\gamma_\alpha-\gamma z_{\alpha\alpha}/z_\alpha)+\frac{z_{\alpha t}}{z_\alpha}\mathcal{C}(\boldsymbol{m})^*. \tag{41}$$

We then take the normal component:

$$\begin{aligned}
\boldsymbol{A}_1\cdot\boldsymbol{n} &= Re\{\mathcal{C}(A_1)^* iz_\alpha/s_\alpha\}\\
&= Re\left\{\frac{iz_\alpha z_{\alpha t}}{2is_\alpha z_\alpha^2}H(\gamma_\alpha-\gamma z_{\alpha\alpha}/z_\alpha)+\frac{iz_\alpha z_{\alpha t}}{s_\alpha z_\alpha}\mathcal{C}(\boldsymbol{m})^*\right\}\\
&= Re\left\{\frac{s_{\alpha t}+is_\alpha\theta_t}{2s_\alpha^2}H(\gamma_\alpha-\gamma i\theta_\alpha)+\frac{i(s_{\alpha t}e^{i\theta}+s_\alpha e^{i\theta}i\theta_t)}{s_\alpha}\mathcal{C}(\boldsymbol{m})^*\right\}\\
&= \frac{s_{\alpha t}}{2s_\alpha^2}H(\gamma_\alpha)+\frac{\theta_t}{2s_\alpha}H(\gamma\theta_\alpha)+\frac{s_{\alpha t}}{s_\alpha}\boldsymbol{m}\cdot\boldsymbol{n}-\theta_t\boldsymbol{m}\cdot\boldsymbol{t}\\
&= \frac{\pi L_t}{L^2}H(\gamma_\alpha)+\frac{\pi\theta_t}{L}H(\gamma\theta_\alpha)+\frac{L_t}{L}\boldsymbol{m}\cdot\boldsymbol{n}-\theta_t\boldsymbol{m}\cdot\boldsymbol{t}.
\end{aligned}$$

We similarly compute the normal component of $\boldsymbol{A}_2$:

$$\begin{aligned}
\boldsymbol{A}_2\cdot\boldsymbol{n} &= Re\{\mathcal{C}(A_2)^* iz_\alpha/s_\alpha\}\\
&= Re\left\{\frac{iz_\alpha^2}{s_\alpha}K[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_{\alpha t}\right)\right\}+Re\left\{\frac{z_\alpha^2}{2s_\alpha}\left[H,\frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_{\alpha t}\right)\right\}+Re\left\{\frac{1}{2s_\alpha}H\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_{\alpha t}\right)\right\}\\
&= Re\left\{\frac{iz_\alpha^2}{s_\alpha}K[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_{\alpha t}\right)\right\}+Re\left\{\frac{z_\alpha^2}{2s_\alpha}\left[H,\frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_{\alpha t}\right)\right\}\\
&+\frac{\pi}{L}H(\gamma_{\alpha t})-\frac{\pi L_t}{L^2}H(\gamma_\alpha)-\frac{\pi}{L}H(\gamma\theta_\alpha\theta_t).
\end{aligned}$$

We reiterate that in this above calculation, use of the formula $z_\alpha = s_\alpha e^{i\theta}$ helps very much to simplify some of the expressions. We continue to rewrite:

$$\boldsymbol{A}_2 \cdot \boldsymbol{n} = Re\left\{\frac{iz_\alpha^2}{s_\alpha}K[z_d]\left(\left(\frac{\gamma_t}{z_\alpha}\right)_\alpha\right)\right\} + Re\left\{\frac{z_\alpha^2}{2s_\alpha}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma_t}{z_\alpha}\right)_\alpha\right)\right\} + Re\left\{\frac{iz_\alpha^2}{s_\alpha}K[z_d]\left(\left(\frac{-\gamma z_{\alpha t}}{z_\alpha^2}\right)_\alpha\right)\right\}$$
$$+ Re\left\{\frac{z_\alpha^2}{2s_\alpha}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{-\gamma z_{\alpha t}}{z_\alpha^2}\right)_\alpha\right)\right\} + \frac{\pi}{L}H(\gamma_{\alpha t}) - \frac{\pi L_t}{L^2}H(\gamma_\alpha) - \frac{\pi}{L}H(\gamma\theta_\alpha\theta_t).$$

To more compactly describe some of the terms on the right-hand side of the $\boldsymbol{A}_2 \cdot \boldsymbol{n}$ equation, we define an operator $S(\gamma_t)$ as follows:

$$S(\gamma_t) = Re\left\{\frac{iz_\alpha^2}{s_\alpha}K[z_d]\left(\left(\frac{\gamma_t}{z_\alpha}\right)_\alpha\right)\right\} + Re\left\{\frac{z_\alpha^2}{2s_\alpha}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma_t}{z_\alpha}\right)_\alpha\right)\right\}. \tag{42}$$

Next, we calculate $\boldsymbol{A}_3 \cdot \boldsymbol{n}$ using the same method as for R_0; that is, we recognize that the squared cosecant term can be written as the derivative of a cotangent, and we integrate by parts. We then write $\boldsymbol{A}_3 \cdot \boldsymbol{n} = \tilde{R}_1 + \tilde{R}_2$ where $\tilde{R}_1$ and $\tilde{R}_2$ are defined as follows:

$$\tilde{R}_1 = Re\left\{\frac{iz_\alpha^2 z_t}{s_\alpha}K[z_d]\left(\frac{(\gamma/z_\alpha)_\alpha}{z_\alpha}\right)_\alpha\right\} - Re\left\{\frac{iz_\alpha^2}{s_\alpha}K[z_d]\left(z_t\left(\frac{(\gamma/z_\alpha)_\alpha}{z_\alpha}\right)_\alpha\right)\right\}$$
$$- Re\left\{\frac{z_\alpha^2}{2s_\alpha}[H, z_t]\left(\frac{1}{z_\alpha}\left(\frac{(\gamma/z_\alpha)_\alpha}{z_\alpha}\right)_\alpha\right)\right\},$$

$$\tilde{R}_2 = Re\left\{\frac{-z_\alpha^2}{2s_\alpha}H\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\frac{z_{\alpha t}}{z_\alpha^2}\right) - \frac{iz_\alpha^2}{s_\alpha}K[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\frac{z_{\alpha t}}{z_\alpha}\right)\right\}$$
$$= Re\left\{\frac{-z_\alpha^2}{2s_\alpha}\left[H, \frac{1}{z_\alpha^2}\right]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha z_{\alpha t}\right) - \frac{iz_\alpha^2}{s_\alpha}K[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\frac{z_{\alpha t}}{z_\alpha}\right)\right\} - Re\left\{\frac{1}{2s_\alpha}H\left(\left(\gamma_\alpha - \gamma\frac{z_{\alpha\alpha}}{z_\alpha}\right)\frac{z_{\alpha t}}{z_\alpha}\right)\right\}.$$

The final term on the right-hand side can be better understood by again making use of the equation $z_\alpha = s_\alpha e^{i\theta}$:

$$\tilde{R}_2 = Re\left\{\frac{-z_\alpha^2}{2s_\alpha}\left[H, \frac{1}{z_\alpha^2}\right]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha z_{\alpha t}\right) - \frac{iz_\alpha^2}{s_\alpha}K[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\frac{z_{\alpha t}}{z_\alpha}\right)\right\} - \frac{\pi L_t}{L^2}H(\gamma_\alpha) - \frac{\pi}{L}H(\gamma\theta_\alpha\theta_t).$$

After all, we say

$$\boldsymbol{W}_{\alpha t} \cdot \boldsymbol{n} = \frac{\pi}{L}H(\gamma_{\alpha t}) + S(\gamma_t) + R_3 + R_4 \tag{43}$$

where

$$\begin{aligned} R_3 = & Re\Big\{\frac{iz_\alpha^2}{s_\alpha}K[z_d]\big(\big(\frac{-\gamma z_{\alpha t}}{z_\alpha^2}\big)_\alpha\big)\Big\} + Re\Big\{\frac{z_\alpha^2}{2s_\alpha}\big[H, \frac{1}{z_\alpha^2}\big]\big(z_\alpha\big(\frac{-\gamma z_{\alpha t}}{z_\alpha^2}\big)_\alpha\big)\Big\} \\ & + Re\Big\{\frac{-z_\alpha^2}{2s_\alpha}\big[H, \frac{1}{z_\alpha^2}\big]\big(\big(\frac{\gamma}{z_\alpha}\big)_\alpha z_{\alpha t}\big) - \frac{iz_\alpha^2}{s_\alpha}K[z_d]\big(\big(\frac{\gamma}{z_\alpha}\big)_\alpha\frac{z_{\alpha t}}{z_\alpha}\big)\Big\} + \tilde{R}_1, \end{aligned} \tag{44}$$

$$R_4 = \frac{\pi\theta_t}{L}H(\gamma\theta_\alpha) - \frac{\pi L_t}{L^2}H(\gamma_\alpha) - \frac{2\pi}{L}H(\gamma\theta_\alpha\theta_t) + \frac{L_t}{L}\boldsymbol{m}\cdot\boldsymbol{n} - \theta_t\boldsymbol{m}\cdot\boldsymbol{t}. \tag{45}$$

3.3 Our small-scale decomposition

We are now going to rewrite the evolution equations to emphasize the terms in the θ_t and γ_t equations which must be treated carefully in the energy estimate.

First, substituting the expressions we have found above for $(\boldsymbol{W}_\alpha \cdot \boldsymbol{t})s_\alpha$ and $\boldsymbol{W}_{\alpha t} \cdot \boldsymbol{n}$ into (26), we immediately get

$$\begin{aligned}
\gamma_t = & -\bar{A}\left(\partial_\alpha^4\theta + \frac{3\theta_\alpha^2\theta_{\alpha\alpha}}{2}\right) + \frac{2\pi(V_W\gamma)_\alpha}{L} \\
& - 2\widetilde{A}\left(\frac{\pi}{L}H(\gamma_{\alpha t}) + S(\gamma_t) + R_3 + R_4\right) - \left(2A - 4\pi\widetilde{A}\theta_\alpha/L\right)(J[z_d](\gamma_t) + R_1 + R_2) \\
& - 2\widetilde{A}\left((V_W)_\alpha\theta_t + V_W\theta_{t\alpha} + \frac{2\pi g x_{\alpha\alpha}}{L}\right) - 2A\left(\left(\frac{\pi^2}{L^2}\right)\gamma\gamma_\alpha - V_W\boldsymbol{W}_\alpha \cdot \boldsymbol{t} + g y_\alpha\right).
\end{aligned}$$

Next, we wish to replace R_2 by substituting from (40) and replace $\boldsymbol{W}_\alpha \cdot \boldsymbol{t}$ from (35):

$$\begin{aligned}
\gamma_t = & -\bar{A}\left(\partial_\alpha^4\theta + \frac{3\theta_\alpha^2\theta_{\alpha\alpha}}{2}\right) + \frac{2\pi(V_W\gamma)_\alpha}{L} \\
& - 2\widetilde{A}\left(\frac{\pi}{L}H(\gamma_{\alpha t}) + S(\gamma_t) + R_3 + R_4\right) - \left(2A - 4\pi\widetilde{A}\theta_\alpha/L\right)(J[z_d](\gamma_t) + R_1) \\
& - \left(2A - 4\pi\widetilde{A}\theta_\alpha/L\right)\left(-\frac{1}{2}H(\gamma\theta_t) - Re\left\{z_\alpha K[z_d]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\} - Re\left\{\frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\}\right) \\
& - 2\widetilde{A}\left((V_W)_\alpha\theta_t + V_W\theta_{t\alpha} + \frac{2\pi g x_{\alpha\alpha}}{L}\right) - 2A\left(\left(\frac{\pi^2}{L^2}\right)\gamma\gamma_\alpha - V_W(-\frac{\pi}{L}H(\gamma\theta_\alpha) + \boldsymbol{m} \cdot \boldsymbol{t}) + g y_\alpha\right).
\end{aligned}$$

Several terms on the right-hand side of this latest expression for γ_t include θ_t; we will be substituting for these using the θ_t equation. To start, since the expression for R_2 contains a term $-\frac{1}{2}H(\gamma\theta_t)$, we use (36) to calculate the following:

$$\begin{aligned}
H(\gamma\theta_t) &= \frac{2\pi^2}{L^2}H(\gamma H(\gamma_\alpha)) + H\left(\frac{2\pi}{L}V_W\theta_\alpha\gamma + \frac{2\pi}{L}\gamma\boldsymbol{m} \cdot \boldsymbol{n}\right) \\
&= \frac{2\pi^2}{L^2}[H, \gamma]H(\gamma_\alpha) - \frac{2\pi^2}{L^2}\gamma\gamma_\alpha + H\left(\frac{2\pi}{L}V_W\theta_\alpha\gamma + \frac{2\pi}{L}\gamma\boldsymbol{m} \cdot \boldsymbol{n}\right).
\end{aligned}$$

Second, all the terms involving θ_t are

$$\begin{aligned}
&\frac{\pi\theta_t}{L}H(\gamma\theta_\alpha) - \frac{2\pi}{L}H(\gamma\theta_\alpha\theta_t) + (V_W)_\alpha\theta_t - \theta_t\boldsymbol{m}\cdot\boldsymbol{t} + V_W\theta_{t\alpha}\\
&= \left(\frac{\pi}{L}H(\gamma\theta_\alpha) + \frac{\pi}{L}H(\gamma\theta_\alpha) - \mathbb{P}(\boldsymbol{m}\cdot\boldsymbol{t}) - \boldsymbol{m}\cdot\boldsymbol{t}\right)\theta_t - \frac{2\pi}{L}H(\gamma\theta_\alpha\theta_t) + V_W\theta_{t\alpha}\\
&= -\frac{2\pi}{L}[H,\theta_t]\gamma\theta_\alpha - (\mathbb{P}(\boldsymbol{m}\cdot\boldsymbol{t}) + (\boldsymbol{m}\cdot\boldsymbol{t}))\theta_t + V_W\theta_{t\alpha}\\
&= -\frac{4\pi^3}{L^3}[H,(H\gamma_\alpha)]\gamma\theta_\alpha - \frac{2\pi}{L}\left[H,\left(\frac{2\pi}{L}V_W\theta_\alpha + \frac{2\pi}{L}\boldsymbol{m}\cdot\boldsymbol{n}\right)\right]\gamma\theta_\alpha\\
&- (\mathbb{P}(\boldsymbol{m}\cdot\boldsymbol{t}) + \boldsymbol{m}\cdot\boldsymbol{t})\left(\frac{2\pi^2}{L^2}H(\gamma_\alpha) + \frac{2\pi}{L}V_W\theta_\alpha + \frac{2\pi}{L}\boldsymbol{m}\cdot\boldsymbol{n}\right)\\
&+ \frac{2\pi^2V_W}{L^2}H(\gamma_{\alpha\alpha}) + \frac{2\pi}{L}V_W^2\theta_{\alpha\alpha} + \frac{2\pi V_W}{L}(V_W)_\alpha\theta_\alpha + \frac{2\pi V_W}{L}(\boldsymbol{m}\cdot\boldsymbol{n})_\alpha\\
&= -\frac{4\pi^3}{L^3}[H,(H\gamma_\alpha)]\gamma\theta_\alpha - (\mathbb{P}(\boldsymbol{m}\cdot\boldsymbol{t}) + \boldsymbol{m}\cdot\boldsymbol{t})\left(\frac{2\pi^2}{L^2}H(\gamma_\alpha)\right) + \frac{2\pi^2V_W}{L^2}H(\gamma_{\alpha\alpha}) + \frac{2\pi}{L}V_W^2\theta_{\alpha\alpha} + R_5,
\end{aligned}$$

where R_5 is

$$\begin{aligned}
R_5 &= -\frac{2\pi}{L}\left[H,\left(\frac{2\pi}{L}V_W\theta_\alpha + \frac{2\pi}{L}\boldsymbol{m}\cdot\boldsymbol{n}\right)\right]\gamma\theta_\alpha - (\mathbb{P}(\boldsymbol{m}\cdot\boldsymbol{t}) + \boldsymbol{m}\cdot\boldsymbol{t})\left(\frac{2\pi}{L}V_W\theta_\alpha + \frac{2\pi}{L}\boldsymbol{m}\cdot\boldsymbol{n}\right)\\
&+ \frac{2\pi}{L}V_W(V_W)_\alpha\theta_\alpha + \frac{2\pi V_W}{L}(\boldsymbol{m}\cdot\boldsymbol{n})_\alpha.
\end{aligned} \tag{46}$$

Substituting all of this, the equation for γ_t is

$$\begin{aligned}
\gamma_t &= -\bar{A}\left(\partial_\alpha^4\theta + \frac{3\theta_\alpha^2\theta_{\alpha\alpha}}{2}\right) + \frac{2\pi((V_W)_\alpha\gamma)}{L} + \frac{2\pi(V_W\gamma_\alpha)}{L}\\
&- 2\widetilde{A}\left(\frac{\pi}{L}H(\gamma_{\alpha t}) + S(\gamma_t)\right) - \left(2A - 4\pi\widetilde{A}\theta_\alpha/L\right)(J[z_d](\gamma_t))\\
&- 2\widetilde{A}\left(-\frac{4\pi^3}{L^2}[H,(H\gamma_\alpha)]\gamma\theta_\alpha - (\mathbb{P}(\boldsymbol{m}\cdot\boldsymbol{t}) + \boldsymbol{m}\cdot\boldsymbol{t})\frac{2\pi^2}{L^2}H(\gamma_\alpha) - \frac{\pi L_t}{L^2}H(\gamma_\alpha) + \frac{2\pi^2}{L^2}V_WH(\gamma_{\alpha\alpha}) + \frac{2\pi}{L}V_W^2\theta_{\alpha\alpha}\right)\\
&+ \left(2A - 4\pi\widetilde{A}\theta_\alpha/L\right)\left(\frac{\pi^2}{L^2}[H,\gamma]H(\gamma_\alpha) - \frac{\pi^2}{L^2}\gamma\gamma_\alpha + H\left(\frac{\pi}{L}V_W\theta_\alpha\gamma + \frac{\pi}{L}\gamma\boldsymbol{m}\cdot\boldsymbol{n}\right)\right)\\
&- \left(2A - 4\pi\widetilde{A}\theta_\alpha/L\right)\left(-Re\left\{z_\alpha K[z_d]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\} - Re\left\{\frac{z_\alpha}{2i}\left[H,\frac{1}{z_\alpha}\right]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\} + R_1\right)\\
&- 2\widetilde{A}\left(\frac{L_t}{L}\boldsymbol{m}\cdot\boldsymbol{n} + R_3 + R_5 + \frac{2\pi g x_{\alpha\alpha}}{L}\right) - 2A\left(\left(\frac{\pi^2}{L^2}\right)\gamma\gamma_\alpha - V_W\left(-\frac{\pi}{L}H(\gamma\theta_\alpha) + \boldsymbol{m}\cdot\boldsymbol{t}\right) + gy_\alpha\right).
\end{aligned}$$

To simplify the notation of the equation for γ_t, we define R and $\tilde{R}$:

$$\begin{aligned}
\widetilde{R} &= -\bar{A}\frac{3\theta_\alpha^2\theta_{\alpha\alpha}}{2} + \frac{2\pi(V_W\gamma_\alpha)}{L} - \left(4A - 4\pi\widetilde{A}\theta_\alpha/L\right)\left(\frac{\pi^2}{L^2}\gamma\gamma_\alpha\right)\\
&-2\widetilde{A}\left(-\frac{4\pi^3}{L^2}[H,(H\gamma_\alpha)]\gamma\theta_\alpha - (\mathbb{P}(\boldsymbol{m}\cdot\boldsymbol{t}) + \boldsymbol{m}\cdot\boldsymbol{t})\frac{2\pi^2}{L^2}H(\gamma_\alpha) - \frac{\pi L_t}{L^2}H(\gamma_\alpha) + \frac{2\pi}{L}V_W^2\theta_{\alpha\alpha}\right),
\end{aligned} \tag{47}$$

$$R = \frac{2\pi((V_W)_\alpha\gamma)}{L} + \left(2A - 4\pi\widetilde{A}\theta_\alpha/L\right)\left(\frac{\pi^2}{L^2}[H,\gamma]H(\gamma_\alpha) + H\left(\frac{\pi}{L}V_W\theta_\alpha\gamma + \frac{\pi}{L}\gamma\boldsymbol{m}\cdot\boldsymbol{n}\right)\right)$$
$$- \left(2A - 4\pi\widetilde{A}\theta_\alpha/L\right)\left(-Re\left\{z_\alpha K[z_d]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\} - Re\left\{\frac{z_\alpha}{2i}\left[H,\frac{1}{z_\alpha}\right]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\} + R_1\right)$$
$$- 2\widetilde{A}\left(\frac{L_t}{L}\boldsymbol{m}\cdot\boldsymbol{n} + \frac{2\pi g x_{\alpha\alpha}}{L} + R_3 + R_5\right) - 2A\left(-V_W\left(-\frac{\pi}{L}H(\gamma\theta_\alpha) + \boldsymbol{m}\cdot\boldsymbol{t}\right) + gy_\alpha\right). \quad (48)$$

Finally, we turn to the operators acting on γ_t. Using the definitions of S and $J[z_d]$, substituting $s_\alpha = L/2\pi$ and rewriting $J[z_d]\gamma_t$, we introduce the operator $T[\theta]$:

$$2\widetilde{A}S(\gamma_t) + \left(2A - 4\pi\widetilde{A}\theta_\alpha/L\right)J[z_d](\gamma_t) = \left(2A - 4\pi\widetilde{A}\theta_\alpha/L\right)Re\left\{z_\alpha K[z_d]\gamma_t + \frac{z_\alpha}{2i}\left[H,\frac{1}{z_\alpha}\right]\gamma_t\right\}$$
$$+ 2\widetilde{A}Re\left\{\frac{2\pi i z_\alpha^2}{L}K[z_d]\left(\left(\frac{\gamma_t}{z_\alpha}\right)_\alpha\right)\right\} + 2\widetilde{A}Re\left\{\frac{\pi z_\alpha^2}{L}\left[H,\frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma_t}{z_\alpha}\right)_\alpha\right)\right\} := T[\theta](\gamma_t). \quad (49)$$

In conclusion, the equation for γ_t is the following:

$$\gamma_t = -\bar{A}\partial_\alpha^4\theta - \frac{2\widetilde{A}\pi}{L}H(\gamma_{\alpha t}) - T[\theta](\gamma_t) - \frac{4\widetilde{A}V_W\pi^2}{L^2}H(\gamma_{\alpha\alpha}) + \widetilde{R} + R. \quad (50)$$

Finally, we group together some like terms, rewriting γ_t as

$$\gamma_t = -\bar{A}\partial_\alpha^4\theta - \frac{2\widetilde{A}\pi}{L}H(\gamma_{\alpha t}) - \frac{4\widetilde{A}V_W\pi^2}{L^2}H(\gamma_{\alpha\alpha}) + Q, \quad (51)$$

where Q is the summation of the smoother terms, $Q = -T[\theta]\gamma_t + \widetilde{R} + R$.

It will be helpful to have a brief notation for the evolution equations, so we introduce the following:

$$(\theta_t, \gamma_t) = (\mathcal{B}_1, \mathcal{B}_2), \quad (52)$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are the right-hand sides of equation (36) and (50) or (51).

4 Preliminary estimates

In this section, we will collect together several estimates which will be useful many times in the sequel. These include interpolation, algebra, and composition estimates for Sobolev spaces, as well as estimates for operators related to the Hilbert transform and the Birkhoff-Rott integral.

First, we have an elementary interpolation lemma. We omit the proof of this lemma; it may be found many places, one of which is [3].

Lemma 4.1. *Let $m \geq 0$ and $s \geq m$ and $f \in H^s$ be given. Then,*

$$\|f\|_m \leq C\|f\|_s^{m/s}\|f\|_0^{1-m/s}. \quad (53)$$

Next, we have the usual algebra property.

Lemma 4.2. *For all $s > 1/2$, $H^s(\mathbb{T})$ is a Banach algebra. That is, there exists $c > 0$ such that for all $u, v \in H^s$,*

$$\|uv\|_s \leq c\|u\|_s\|v\|_s. \tag{54}$$

We next have an elementary composition estimate; see, for instance, [25].

Lemma 4.3. *If F is a smooth function and u is in $H^k \cap L^\infty$, then $\|F(u)\|_k \leq C(1 + \|u\|_k)$. The constant C depends on $|F^{(j)}(u)|_{L^\infty}$, for $0 \leq j \leq k$.*

As in other works in the field such as [30], we must ensure that the curves z we consider are non-self-intersecting. Following [30], we use the chord-arc condition for this purpose. Specifying that z is non-self-intersecting thus means that we require that it satisfies the following estimate for some $\bar{c} > 0$ and for all α and α':

$$|q_1| = \left|\frac{z_d(\alpha) - z_d(\alpha')}{\alpha - \alpha'}\right| > \bar{c}. \tag{55}$$

We now turn to estimates for our integral operator K, which is the remainder from approximating the Birkhoff-Rott integral with a Hilbert transform.

Lemma 4.4. *Let $n \geq 3$ be an integer. Assume $z_d \in H^n$ and there exists $\beta > 0$, such that for all α, α',*

$$\left|q_1[z_d](\alpha, \alpha')\right| > \beta \tag{56}$$

Then $K[z_d] : H^1 \to H^{n-1}$ and $K[z_d] : H^0 \to H^{n-2}$, with the following estimates:

$$\|K[z_d]f\|_{n-1} \leq C\|f\|_1 \exp\{C_1\|z_d\|_n\}, \tag{57}$$

$$\|K[z_d]f\|_{n-2} \leq C\|f\|_0 \exp\{C_1\|z_d\|_n\}. \tag{58}$$

Furthermore,

$$\|K[z_d](f_\alpha)\|_{n-3} \leq C\|f\|_0 \exp\{C_1\|z_d\|_n\}. \tag{59}$$

We again do not include the proof here; instead, the interested reader might consult [3]. We also will need a Lipschitz estimate for operator K; this will be useful when invoking the Picard Theorem during our existence proof, and also when establishing uniqueness and continuous dependence of solutions.

Lemma 4.5. *Let θ and θ' be in H^3. Let L and L' be the corresponding lengths of the associated curves z_d and z_d' and let q_1 and q_1' be the associated chord-arc quantities. Assume there exist positive constants β_1 and β_2 such that $L < \beta_2$ and $L' < \beta_2$ and for all α and α',*

$$|q_1(\alpha, \alpha')| > \beta_1, \quad |q_1'(\alpha, \alpha')| > \beta_1.$$

Then the following Lipschitz estimate holds, for any $f \in H^1$

$$\|K[z_d]f - K[z_d']f\|_1 \leq C\|\theta - \theta'\|_1\|f\|_1. \tag{60}$$

See, for instance, [2], [3] or [5] for proof and discussion of this lemma.

We have a few different commutator estimates for the commutator of the Hilbert transform and multiplication by a smooth function. These balance how much regularity we find for the commutator against how much regularity we require on the function being acted upon.

Lemma 4.6. *Let $n \geq 1$ be an integer. Let $\phi \in H^n$ be given. Then $[H, f] : H^0 \to H^{n-1}$ and $[H, \phi] : H^{-1} \to H^{n-2}$*

$$\|[H, \phi]f\|_{n-1} \leq c\|\phi\|_n\|f\|_0, \tag{61}$$

$$\|[H, \phi]f\|_{n-2} \leq c\|\phi\|_n\|f\|_{-1}. \tag{62}$$

Furthermore, if $j \geq 1$ and $n \geq 2j$ then $[H, \phi] : H^{n-j} \to H^n$

$$\|[H, \phi]f\|_n \leq c\|\phi\|_n\|f\|_{n-j}. \tag{63}$$

We again omit the proof, and again refer the reader to [3] and [5], as well as the earlier work [10], for the proofs of these estimates.

5 Existence

Before proving existence of solutions, we first must introduce a regularized system of evolution equations. We will first prove existence of solutions for the regularized system, and then prove energy estimates for the regularized system. We will then be able to pass to the limit as the regularization parameter vanishes, finding that solutions of the non-regularized system of evolution equations exist.

5.1 The mollified system

We will need to be careful about reconstructing a curve from a tangent angle; this is because not every periodic tangent angle function will lead to a periodic curve. In particular, say η is our tangent angle function, perhaps at a step of an iteration procedure (so that η cannot be assumed to be a solution of our evolution equation). First we concern ourselves with defining the length of the curve; this comes from the horizontal periodicity. The derivative of the horizontal component of the curve to be constructed from η is

$$x_\alpha[\eta, L] = \frac{L}{2\pi}\cos(\eta). \tag{64}$$

To solve for L by the horizontal periodicity requirement $x[\eta, L](2\pi) - x[\eta, L](0) = 2\pi$, we integrate (64) and solve, finding

$$L = \frac{4\pi^2}{\int_0^{2\pi}\cos(\eta(\alpha))d\alpha}. \tag{65}$$

One important immediate result we can get from the above equality is

$$L \geq 2\pi.$$

The vertical periodicity requires $\int_0^{2\pi} y_\alpha[\eta, L](\alpha)d\alpha = y[\eta, L](2\pi) - y[\eta, L](0) = 0$. We define $y_\alpha[\eta, L]$ to be

$$y_\alpha[\eta] = \frac{L[\eta]}{2\pi}\left(\sin(\eta) - \frac{1}{2\pi}\int_0^{2\pi}\sin(\eta)d\alpha\right) = \frac{L[\eta]}{2\pi}\mathbb{P}\sin(\eta).$$

Now we will define the mollified curve first. We let θ^ε be given, and we define the length L^ε as

$$L^\varepsilon = L[\theta^\varepsilon].$$

Naturally, since $\bar{A} = \frac{8\pi^3 E_b}{(\rho_1+\rho_2)L^3}$ and $\widetilde{A} = \frac{2\pi\rho_0}{(\rho_1+\rho_2)L}$, we will define $\bar{A}^\varepsilon$ and $\widetilde{A}^\varepsilon$ as

$$\bar{A}^\varepsilon = \frac{8\pi^3 E_b}{(\rho_1+\rho_2)(L^\varepsilon)^3}, \quad \widetilde{A}^\varepsilon = \frac{2\pi\rho_0}{(\rho_1+\rho_2)L^\varepsilon}. \tag{66}$$

The derivative of the curve is given by

$$x_\alpha^\varepsilon = x_\alpha[\theta^\varepsilon, L^\varepsilon] = \frac{L^\varepsilon}{2\pi}\cos(\theta^\varepsilon), \tag{67}$$

$$y_\alpha^\varepsilon = y_\alpha[\theta^\varepsilon, L^\varepsilon] = \frac{L^\varepsilon}{2\pi}\mathbb{P}\sin(\theta^\varepsilon). \tag{68}$$

The mollified curve is then defined by integrating:

$$z_d^\varepsilon = \frac{L^\varepsilon}{2\pi}\int_0^\alpha \cos(\theta^\varepsilon) + i\mathbb{P}\sin(\theta^\varepsilon)\, d\alpha, \tag{69}$$

and the unit normal and the tangent vectors are defined to be

$$\boldsymbol{t}^\varepsilon = (\cos(\theta^\varepsilon), \sin(\theta^\varepsilon)), \quad \boldsymbol{n}^\varepsilon = (-\sin(\theta^\varepsilon), \cos(\theta^\varepsilon)). \tag{70}$$

Note that if the mean of $\sin(\theta^\varepsilon)$ is not equal to zero, then these vector $\boldsymbol{t}^\varepsilon, \boldsymbol{n}^\varepsilon$ are not actually the unit tangent and normal vectors to the curve z_d^ε. We will ensure that when θ^ε is a solution of evolution equation, the equation $\frac{1}{2\pi}\int_0^{2\pi}\sin(\theta^\varepsilon)d\alpha = 0$ holds.

Now we study $\int_0^{2\pi}\sin(\theta(\alpha, t))d\alpha$. By the evolution equation $\theta_t = \frac{U_\alpha + V\theta_\alpha}{s_\alpha}$, we have

$$\frac{d}{dt}\int_0^{2\pi}\sin(\theta(\alpha, t))d\alpha = \frac{1}{s_\alpha}\int_0^{2\pi}\cos(\theta(\alpha, t))(U_\alpha + V\theta_\alpha)d\alpha. \tag{71}$$

Recall that $V_\alpha = \frac{L_t}{2\pi} + \theta_\alpha U$; we integrate by parts in the above integral:

$$\begin{aligned}
&\frac{d}{dt}\int_0^{2\pi} \sin(\theta(\alpha,t))\, d\alpha = \frac{1}{s_\alpha}\int_0^{2\pi} \sin(\theta(\alpha,t))\theta_\alpha U\, d\alpha - \frac{1}{s_\alpha}\int_0^{2\pi} V_\alpha \sin(\theta(\alpha,t))\, d\alpha \\
&= \frac{1}{s_\alpha}\int_0^{2\pi} \sin(\theta(\alpha,t))\theta_\alpha U\, d\alpha - \frac{1}{s_\alpha}\int_0^{2\pi} \left(\frac{L_t}{2\pi} + \theta_\alpha U\right)\sin(\theta(\alpha,t))\, d\alpha \\
&= -\frac{L_t}{L}\int_0^{2\pi} \sin(\theta(\alpha,t))\, d\alpha.
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\left(L\int_0^{2\pi} \sin(\theta(\alpha,t))\, d\alpha\right) = 0. \tag{72}$$

Thus for a solution θ of the exact evolution equations, we see that if the mean of $\sin(\theta)$ is initially zero, then it will remain zero at positive times.

We introduce the following analogue of (52) for the mollified system:

$$(\theta^\varepsilon, \gamma^\varepsilon)_t = (\mathcal{B}_1^\varepsilon + \nu^\varepsilon, \mathcal{B}_2^\varepsilon), \tag{73}$$

where $\mathcal{B}_1^\varepsilon$, ν^ε, and $\mathcal{B}_2^\varepsilon$ are to be defined now. Of course, $\mathcal{B}_1^\varepsilon, \mathcal{B}_2^\varepsilon$ will be rather clearly similar to $\mathcal{B}_1$, $\mathcal{B}_2$, respectively. The additional term ν^ε will be to enforce our periodicity requirement (that the mean of $\sin(\theta^\epsilon)$ is zero). Here ν^ε is taken to be a function of t and to be independent of α. By the system (73),

$$\frac{d}{dt}\int_0^{2\pi} \sin(\theta^\varepsilon)\, d\alpha = \int_0^{2\pi} \mathcal{B}_1^\varepsilon \cos(\theta^\varepsilon)\, d\alpha + \nu^\varepsilon \int_0^{2\pi} \cos(\theta^\varepsilon)\, d\alpha. \tag{74}$$

As we have discussed before, we need $\frac{d}{dt}\int_0^{2\pi} \sin(\theta^\varepsilon)\, d\alpha = 0$. So from (65) we define ν^ε as

$$\nu^\varepsilon = -\frac{\int_0^{2\pi} \mathcal{B}_1^\varepsilon \cos(\theta^\varepsilon) d\alpha}{\int_0^{2\pi} \cos(\theta^\varepsilon) d\alpha} = -\frac{L^\varepsilon}{4\pi^2}\int_0^{2\pi} \mathcal{B}_1^\varepsilon \cos(\theta^\varepsilon)\, d\alpha. \tag{75}$$

The point of this is that if $(\theta^\varepsilon, \gamma^\varepsilon)$ solves the system (73), then the mean of $\sin(\theta^\varepsilon(\alpha,t))$ equals zero. In light of (67), (68), this implies

$$|z_\alpha^\varepsilon| = \frac{L^\varepsilon}{2\pi}, \quad \forall \alpha,$$

as desired; this would not be the case if the mean of $\sin(\theta^\varepsilon)$ were nonzero.

5.1.1 An assumption on the solvability of the integral equation

We will use the Picard theorem to prove existence of solutions for the initial value problem for the mollified system. In order to use it, we need to introduce the following open set $\mathcal{O}$. We let s be a sufficiently large integer. Let positive constants $\bar{d}_1, \bar{d}_2$ and $\bar{d}_3$ be given. Let $\mathcal{O}$ be the subset of $H^s \times H^{s-1}$ such that for all $(f_1, f_2) \in \mathcal{O}$, the following three conditions are satisfied:

$$\|(f_1, f_2)\|_{H^s \times H^{s-1}} < \bar{d}_1, \tag{76}$$
$$L[f_1] < \bar{d}_2, \tag{77}$$
$$|q_1[f_1](\alpha, \alpha')| > \bar{d}_3, \quad \forall \alpha, \alpha'. \tag{78}$$

We now consider $(\theta, \gamma) \in \mathcal{O}$. Before mollifying the equations, it is necessary to solve for γ_t in the open set. The operator $T[\theta]$ involves z_d and z_α, so we first give the estimates on z_d and z_α in terms of θ.

Lemma 5.1. *Let $(\theta^\varepsilon, \gamma^\varepsilon) \in \mathcal{O}$ be given. Then, the following estimates are satisfied:*

$$\|z^\varepsilon_\alpha\|_s \leq C(1 + \|\theta^\varepsilon\|_s), \tag{79}$$
$$\|z^\varepsilon_d\|_{s+1} \leq C(1 + \|\theta^\varepsilon\|_s). \tag{80}$$

Proof. The inequality of (79) follows immediately from the equation (67) and (68), together with the standard composition estimate (Lemma 4.3) and the fact that the definition of $\mathcal{O}$ includes a bound on the length. Since z^ε_d is defined in (69) by an integral of $x^\varepsilon_\alpha, y^\varepsilon_\alpha$, the estimate (80) follows. □

The equation (49) gives definition of $T[\theta](\gamma_t)$; generally for any $f \in H^0$, we have the definition

$$T[\theta](f) = \big(2A - 4\pi\widetilde{A}\theta_\alpha/L\big) Re\left\{ z_\alpha K[z_d]f + \frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha}\right] f \right\} + 2\widetilde{A} Re\left\{ \frac{2\pi i z_\alpha^2}{L} K[z_d]\left(\left(\frac{f}{z_\alpha}\right)_\alpha\right)\right\} + 2\widetilde{A} Re\left\{ \frac{\pi z_\alpha^2}{L}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{f}{z_\alpha}\right)_\alpha\right)\right\}. \tag{81}$$

Lemma 5.2. *Let $(\theta, \gamma) \in \mathcal{O}$ be given. Then $T[\theta]$ is a bounded operator from H^0 to H^0. Moreover, there exist positive constants C_1 and C_2 such that for any $f \in H^0$,*

$$\|T[\theta]f\|_0 \leq C_1(A + \tilde{A}_0)\exp\{C_2\|\theta\|_2\}\|f\|_0, \tag{82}$$

where $\tilde{A}_0 = \dfrac{\rho_0}{\rho_1 + \rho_2}$.

Proof. The operator contains $K[z_d]$ and Hilbert commutators. The following estimates take them into account separately.

By Lemma 4.4, for $n \geq 3$, we have the following estimates:

$$\|K[z_d]f\|_0 \leq C_1\|f\|_0 \exp\{C_2\|z_d\|_2\},$$

and

$$\left\| K[z_d]\left(\left(\frac{f}{z_\alpha}\right)_\alpha\right)\right\|_0 \le C_1 \left\|\left(\frac{f}{z_\alpha}\right)\right\|_0 \exp\{C_2\|z_d\|_3\} \le C_1\|f\|_0 \exp\{C_2\|z_d\|_3\}.$$

By Lemma 4.6, we have the following:

$$\left\|\left[H, \frac{1}{z_\alpha^2}\right] f\right\|_0 \le C_1 \left\|\frac{1}{z_\alpha^2}\right\|_1 \|f\|_0 \le C_1\|f\|_0 \exp\{C_2\|z_\alpha\|_1\},$$

and

$$\left\|\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{f}{z_\alpha}\right)_\alpha\right)\right\|_0 \le C_1 \left\|\frac{1}{z_\alpha^2}\right\|_2 \left\|\left(z_\alpha\left(\frac{f}{z_\alpha}\right)_\alpha\right)\right\|_{-1} \le C_1\|f\|_0 \exp\{C_2\|z_\alpha\|_2\}.$$

We now conclude that by Lemma 4.2 and Lemma 5.1, $T[\theta]$ is an operator from H^0 to H^0, moreover,

$$\|T[\theta]f\|_0 \le C_1\left(A + \tilde{A}\right)\exp\{C_2\|\theta\|_2\}\|f\|_0.$$

(Note that the constants C_1 and C_2 may depend upon the constants $\bar{d}_1$, $\bar{d}_2$, and $\bar{d}_3$ which define our open set, but this causes no problem.) Since $\tilde{A} = 2\pi\tilde{A}_0/L \le \tilde{A}_0$, this completes the proof of the lemma. □

Generally, using Lemmas 4.2, 4.4 , 4.6 and Lemma 5.1 again for general s, the following estimate holds:

$$\|T[\theta]f\|_{s-2} \le C_1(A + \tilde{A}_0)\exp\{C_2\|\theta\|_s\}\|f\|_0. \tag{83}$$

The following lemma on the resolvent appears as Proposition 9.6 of [20]:

Lemma 5.3. *If A is a bounded linear operator on a Hilbert space, then the resolvent set $\rho(A)$ is an open subset of $\mathbb{C}$ that contains the exterior disc $\{\lambda \in \mathbb{C}, \quad |\lambda| > \|A\|\}$. The resolvent $R_\lambda = (\lambda I - A)^{-1}$ is an operator-valued analytic function of λ defined on $\rho(A)$. Moreover*

$$\|(\lambda I - A)^{-1}\| \le \frac{1}{|\lambda| - \|A\|}.$$

We will assume that the integral equation is solvable, and prove well-posedness of our initial value problem under this assumption. It is important to note that by Lemma 5.3, we can verify that the assumption is indeed satisfied at least for certain values of ρ_0, ρ_1, and ρ_2. For example, we can see that the operator $T[\theta]$ is invertible as desired if ρ_0 is small.

Assumption 5.4. *For fixed ρ_0, ρ_1, and ρ_2, and for all $(\theta, \gamma) \in \mathcal{O}$, the operator $T[\theta]$ satisfies*

$$\|T[\theta]\|_{H^0 \to H^0} \le c \tag{84}$$

for some constant $c < 1$.

Notice that the Fourier multiplier operator $I + \frac{2\pi}{L}\widetilde{A}\Lambda$ is invertible and $\left\|(I + \frac{2\pi}{L}\widetilde{A}\Lambda)^{-1}\right\|_{H^s \to H^s} \leq 1$. Thus, we may conclude

$$\left\| \left(I + \frac{2\pi}{L}\widetilde{A}\Lambda\right)^{-1} T[\theta] \right\|_{H^0 \to H^0} \leq c. \tag{85}$$

By Lemma 5.3, the inverse $\big(I + \big(I + \frac{2\pi}{L}\widetilde{A}\Lambda\big)^{-1} T[\theta]\big)^{-1}$ exists and is a uniformly bounded operator from H^0 to H^0 for all $(\theta, \gamma) \in \mathcal{O}$.

5.1.2 Mollified Equations

We introduce the mollifier χ_ε with the parameter ε: this operator acts through truncation of the Fourier series, zeroing out modes with wave number larger that $1/\varepsilon$. As such, χ_ε is a projection, so that $\chi_\varepsilon^2 = \chi_\varepsilon$. We now define $\mathcal{B}_1^\varepsilon, \mathcal{B}_2^\varepsilon$. To begin, we make the following definitions:

$$\mathcal{B}_1^\varepsilon = \frac{2\pi^2}{(L^\varepsilon)^2}\chi_\varepsilon H(\gamma_\alpha^\varepsilon) + \frac{2\pi}{L^\varepsilon}\chi_\varepsilon\left(V_W^\varepsilon(\chi_\varepsilon\theta_\alpha)\right) + \frac{2\pi}{L^\varepsilon}\boldsymbol{m}^{\boldsymbol{\varepsilon}} \cdot \boldsymbol{n}^{\boldsymbol{\varepsilon}}, \tag{86}$$

$$\mathcal{B}_2^\varepsilon = -\bar{A}^\varepsilon\chi_\varepsilon\partial_\alpha^4\theta^\varepsilon - \frac{2\widetilde{A}^\varepsilon\pi}{L^\varepsilon}H(\gamma_{\alpha t}^\varepsilon) - \frac{4\widetilde{A}^\varepsilon\pi^2}{(L^\varepsilon)^2}\chi_\varepsilon(V_W^\varepsilon\chi_\varepsilon H(\gamma_{\alpha\alpha}^\varepsilon)) + Q^\varepsilon. \tag{87}$$

The mollification operator χ_ε appears twice in some of terms to allow us to perform integration by parts in the energy estimate. In the mollified system, there remain several terms to define, and we will do this presently.

We define $\boldsymbol{m}^\varepsilon$ the same way that $\boldsymbol{m}$ is defined in (34), but we use the mollified quantities instead:

$$\mathcal{C}(\boldsymbol{m}^\varepsilon)^* = z_\alpha^\varepsilon K[z_d^\varepsilon]\left(\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)_\alpha\right) + \frac{z_\alpha^\varepsilon}{2i}\left[H, \frac{1}{(z_\alpha^\varepsilon)^2}\right]\left(z_\alpha^\varepsilon\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)_\alpha\right). \tag{88}$$

Then from equation (37), we define V_W^ε as follows:

$$V_W^\varepsilon = \partial_\alpha^{-1}\left(\frac{\pi}{L^\varepsilon}H((\chi_\varepsilon\gamma)(\chi_\varepsilon\theta_\alpha^\varepsilon)) - \mathbb{P}(\boldsymbol{m}^\varepsilon \cdot \boldsymbol{t}^\varepsilon)\right). \tag{89}$$

Similarly, the mollified Birkhoff-Rott integral, $\boldsymbol{W}^\varepsilon$, is defined the same way as $\boldsymbol{W}$, using (28), but in terms of the new quantities $\gamma^\varepsilon, z_d^\varepsilon$:

$$\mathcal{C}(\boldsymbol{W}^\varepsilon)^* = \frac{1}{4\pi i}\int_0^{2\pi}\gamma^\varepsilon(\alpha')\cot\left(\frac{1}{2}(z_d^\varepsilon(\alpha) - z_d^\varepsilon(\alpha')\right)d\alpha'. \tag{90}$$

Then, we define U^ϵ to be

$$U^\varepsilon = \boldsymbol{W}^\varepsilon \cdot \boldsymbol{n}^\varepsilon. \tag{91}$$

We also need the mollified version of V, which is as defined in (17):

$$V^\varepsilon = \partial_\alpha^{-1}\mathbb{P}(\theta_\alpha^\varepsilon U^\varepsilon) + V_0^\varepsilon(t). \tag{92}$$

where $V_0^\varepsilon(t)$ is defined by

$$V_0^\varepsilon(t) = \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{W}^\varepsilon \cdot \boldsymbol{t}^\varepsilon d\alpha. \tag{93}$$

According to the definition of $\mathcal{B}_2^\varepsilon$, we still need to define Q^ε. To define it, we need to rewrite the γ_t equation. We rewrite it as

$$\gamma_t = -\frac{2\widetilde{A}\pi}{L}H(\gamma_{\alpha t}) - T[\theta](\gamma_t) + F, \tag{94}$$

where

$$F = -\bar{A}\partial_\alpha^4\theta - \frac{4\widetilde{A}V_W\pi^2}{L^2}H(\gamma_{\alpha\alpha}) + \widetilde{R} + R. \tag{95}$$

By the assumption, for all $(\theta,\gamma)\in\mathcal{O}$, we can solve for γ_t:

$$\gamma_t = \left(I + \left(I + \frac{2\pi}{L}\widetilde{A}\Lambda\right)^{-1} T[\theta]\right)^{-1}\left(I + \frac{2\pi}{L}\widetilde{A}\Lambda\right)^{-1} F. \tag{96}$$

Then, we can rewrite Q as follows:

$$Q = -T[\theta]\left(\left(I + \left(I + \frac{2\pi}{L}\widetilde{A}\Lambda\right)^{-1} T[\theta]\right)^{-1}\left(I + \frac{2\pi}{L}\widetilde{A}\Lambda\right)^{-1} F\right) + \widetilde{R} + R. \tag{97}$$

Now we will define F^ε, R^ε, and $\tilde{R}^\varepsilon$..

From (95), we define F^ε as

$$F^\varepsilon = -\bar{A}^\varepsilon\partial_\alpha^4\theta^\varepsilon - \frac{4\widetilde{A}^\varepsilon V_W^\varepsilon\pi^2}{(L^\varepsilon)^2}H(\gamma_{\alpha\alpha}^\varepsilon) + \tilde{R}^\varepsilon + R^\varepsilon. \tag{98}$$

This still leaves us needing to define R^ε and $\tilde{R}^\varepsilon$. Since R and $\tilde{R}$ involve L_t and $z_t, z_{\alpha t}$, we now define their mollified versions, and then replace all corresponding terms in R^ε and $\widetilde{R}^\varepsilon$.

$$L_t^\varepsilon = -\int_0^{2\pi}\theta_\alpha^\varepsilon U^\varepsilon d\alpha, \qquad z_t^\varepsilon = \mathcal{C}(U^\varepsilon\boldsymbol{n}^\varepsilon + V^\varepsilon\boldsymbol{t}^\varepsilon), \text{ and } \qquad z_{\alpha t}^\varepsilon = \partial_\alpha\mathcal{C}(U^\varepsilon\boldsymbol{n}^\varepsilon + V^\varepsilon\boldsymbol{t}^\varepsilon). \tag{99}$$

The definition of $\tilde{R}^\varepsilon$ is then straightforward:

$$\begin{aligned}\widetilde{R}^\varepsilon &= -\bar{A}^\varepsilon\frac{3(\theta_\alpha^\varepsilon)^2\theta_{\alpha\alpha}^\varepsilon}{2} - 2\widetilde{A}^\varepsilon\left(-\frac{4\pi^3}{(L^\varepsilon)^2}[H,(H\gamma_\alpha^\varepsilon)]\gamma^\varepsilon\theta_\alpha^\varepsilon - (\mathbb{P}(\boldsymbol{m}^\varepsilon\cdot\boldsymbol{t}^\varepsilon) + \boldsymbol{m}^\varepsilon\cdot\boldsymbol{t}^\varepsilon)\frac{2\pi^2}{(L^\varepsilon)^2}H(\gamma_\alpha^\varepsilon)\right)\\ &-2\widetilde{A}^\varepsilon\left(\frac{\pi\int_0^{2\pi}\theta_\alpha^\varepsilon U^\varepsilon d\alpha}{(L^\varepsilon)^2}H(\gamma_\alpha^\varepsilon) + \frac{2\pi}{L^\varepsilon}(V_W^\varepsilon)^2\theta_{\alpha\alpha}^\varepsilon\right) + \frac{2\pi(V_W^\varepsilon\gamma_\alpha^\varepsilon)}{L^\varepsilon} - \left(4A - 4\pi\widetilde{A}^\varepsilon\theta_\alpha^\varepsilon/L^\varepsilon\right)\left(\frac{\pi^2}{(L^\varepsilon)^2}\gamma^\varepsilon\gamma_\alpha^\varepsilon\right).\end{aligned} \tag{100}$$

We omit the details of definition of R^ε, which is similar to the definition of R, but in terms of all the new quantities we have defined. Now we write out the formula for the operator $T[\theta^\varepsilon]$ in terms of the mollified quantities from the definition (81)

$$
\begin{aligned}
T[\theta^\varepsilon](f) &= \left(2A - 4\pi\widetilde{A}^\varepsilon\theta^\varepsilon_\alpha/L^\varepsilon\right) Re\left\{ z^\varepsilon_\alpha K[z^\varepsilon_d]f + \frac{z^\varepsilon_\alpha}{2i}\left[H, \frac{1}{z^\varepsilon_\alpha}\right] f\right\} \\
&+ 2\widetilde{A}^\varepsilon Re\left\{\frac{2\pi i (z^\varepsilon_\alpha)^2}{L^\varepsilon} K[z^\varepsilon_d]\left(\left(\frac{f}{z^\varepsilon_\alpha}\right)_\alpha\right)\right\} + 2\widetilde{A}^\varepsilon Re\left\{\frac{\pi(z^\varepsilon_\alpha)^2}{L^\varepsilon}\left[H, \frac{1}{(z^\varepsilon_\alpha)^2}\right]\left(z^\varepsilon_\alpha\left(\frac{f}{z^\varepsilon_\alpha}\right)_\alpha\right)\right\}. \quad (101)
\end{aligned}
$$

Finally, Q^ε is defined by

$$
Q^\varepsilon = -T[\theta^\varepsilon]\left(\left(I + \left(I + \frac{2\pi}{L^\varepsilon}\widetilde{A}^\varepsilon\Lambda\right)^{-1} T[\theta^\varepsilon]\right)^{-1}\left(I + \frac{2\pi}{L^\varepsilon}\widetilde{A}^\varepsilon\Lambda\right)^{-1} F^\varepsilon\right) + \tilde{R}^\varepsilon + R^\varepsilon. \quad (102)
$$

5.2 Some *A Priori* Estimates

In this section, we give estimates for quantities like $\boldsymbol{W}^\varepsilon$ or V^ε_W, in terms of norms of θ^ε and γ^ε . We note that the same estimates apply when there is no regularization, i.e., when $\varepsilon = 0$, and the proof is exactly the same.

Lemma 5.5. *Let $(\theta^\varepsilon, \gamma^\varepsilon) \in \mathcal{O}$ be given, such that θ^ε satisfies $\langle\langle \sin(\theta^\varepsilon)\rangle\rangle = 0$. Then, the following estimates are satisfied:*

$$
\begin{aligned}
&\|\boldsymbol{t}^\varepsilon\|_s \le C(1 + \|\theta^\varepsilon\|_s), && (103)\\
&\|\boldsymbol{n}^\varepsilon\|_s \le C(1 + \|\theta^\varepsilon\|_s), && (104)\\
&\|\boldsymbol{m}^\varepsilon\|_s \le C_1\|\gamma^\varepsilon\|_{s-1}\exp(C_2\|\theta^\varepsilon\|_s), && (105)\\
&\|\boldsymbol{W}^\varepsilon\|_{s-1} \le C_1\|\gamma^\varepsilon\|_{s-1}\exp(C_2\|\theta^\varepsilon\|_s), && (106)\\
&\|U^\varepsilon\|_{s-1} \le C_1\|\gamma^\varepsilon\|_{s-1}\exp(C_2\|\theta^\varepsilon\|_s), && (107)\\
&|L^\varepsilon_t| \le C_1\|\gamma^\varepsilon\|_{s-1}\exp(C_2\|\theta^\varepsilon\|_s), && (108)\\
&\|V^\varepsilon_W\|_s \le C_1\|\gamma^\varepsilon\|_{s-1}\exp(C_2\|\theta^\varepsilon\|_s), && (109)\\
&\|V^\varepsilon\|_s \le C_1\|\gamma^\varepsilon\|_{s-1}\exp(C_2\|\theta^\varepsilon\|_s), && (110)\\
&\|z^\varepsilon_t\|_{s-1} \le C_1\|\gamma^\varepsilon\|_{s-1}\exp(C_2\|\theta^\varepsilon\|_s), && (111)\\
&\|R^\varepsilon\|_{s-1} \le C_1(1 + \|\gamma^\varepsilon\|^2_{s-1})\exp(C_2\|\theta^\varepsilon\|_s), && (112)\\
&\|\tilde{R}^\varepsilon\|_{s-2} \le C_1(1 + \|\gamma^\varepsilon\|^2_{s-1})\exp(C_2\|\theta^\varepsilon\|_s), && (113)\\
&\|Q^\varepsilon\|_{s-2} \le C_1(1 + \|\gamma^\varepsilon\|^2_{s-1})\exp(C_2\|\theta^\varepsilon\|_s), && (114)\\
&|\nu^\varepsilon| \le C_1\|\gamma^\varepsilon\|_{s-1}\exp(C_2\|\theta^\varepsilon\|_s). && (115)
\end{aligned}
$$

Proof. We will use Lemma 5.1 in the following proof when we meet z^ε_α and z^ε_d. The estimates (103) and (104) follow immediately from (70), together with the standard composition estimate, Lemma 4.3. By the definition of $\boldsymbol{m}^\varepsilon$ and Lemma 4.2, Lemma 4.4, and Lemma 4.6, we have

$$\begin{aligned}\|\boldsymbol{m}^\varepsilon\|_s &\leq \|z_\alpha^\varepsilon\|_s \left\| K[z_d^\varepsilon]\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)_\alpha \right\|_s + \left\|\frac{z_\alpha}{2i}\right\|_s \left\| \left[H, \frac{1}{(z_\alpha^\varepsilon)^2}\right]\left(z_\alpha^\varepsilon\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)_\alpha\right)\right\|_s \\ &\leq C_1\|z_\alpha^\varepsilon\|_s \left\|\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)_\alpha\right\|_1 \exp\{C_2\|z_d^\varepsilon\|_{s+1}\} + \left\|\frac{z_\alpha^\varepsilon}{2i}\right\|_s \left\|\frac{1}{(z_\alpha^\varepsilon)^2}\right\|_s \left\| z_\alpha^\varepsilon\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)_\alpha\right\|_{s-2} \\ &\leq C_1\|\gamma^\varepsilon\|_{s-1}\exp(C_2\|\theta^\varepsilon\|_s).\end{aligned}$$

By the definition of $\boldsymbol{W}^\varepsilon$, and adding and subtracting, we rewrite it as

$$\mathcal{C}(\boldsymbol{W}^\varepsilon)^* = K[z_d^\varepsilon]\gamma^\varepsilon + \frac{1}{2i}H\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right). \tag{116}$$

Lemma 4.4 provides a bound for $K[z_d^\varepsilon]$:

$$\|\boldsymbol{W}^\varepsilon\|_{s-1} \leq \|K[z_d^\varepsilon]\gamma^\varepsilon\|_{s-1} + \left\|\frac{1}{2i}H\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)\right\|_{s-1} \leq C_1\|\gamma^\varepsilon\|_{s-1}\exp(C_2\|\theta^\varepsilon\|_s). \tag{117}$$

The estimate (107) immediately follows from (91) and the bounds (104) and (117). The estimate (108) follows from (99), (107) and the Schwartz inequality. The estimate on V_W^ε readily follows from its definition (89) and the bound on $\boldsymbol{m}^\varepsilon$. To estimate V^ε, we first do $V_0^\varepsilon(t)$. By the definition (93), $|V_0^\varepsilon(t)|$ is bounded from (103) and (106) and the Schwartz inequality. Then the estimate on V^ε is straightforward from its definition (92) and the bound on U^ε. The estimate (111) is obtained easily from the definition $z_t^\varepsilon = \mathcal{C}(U^\varepsilon\boldsymbol{n}^\varepsilon + V^\varepsilon\boldsymbol{t}^\varepsilon)$ and previous bounds.

To get the estimate on R^ε, we first obtain bounds on terms which involving the operator $K[z_d^\varepsilon]$ and all kinds of Hilbert commutators. They are smoothing enough for sufficiently large s by Lemma 4.4 and Lemma 4.6. And then all the terms comprising R^ε are bounded in H^{s-1} from Lemma 4.2 and all previous estimates. This demonstrates the inequality (112).

We have defined $\tilde{R}^\varepsilon$ in (100), and it contains as the highest derivative terms $\gamma_\alpha^\varepsilon$ and $\theta_{\alpha\alpha}^\varepsilon$. So the inequality (113) follows from Lemma 4.2 and all previous estimates.

Now we turn to estimate Q^ε. Firstly $\|F^\varepsilon\|_0 \leq C_1(1+\|\gamma^\varepsilon\|_{s-1}^2)\exp(C_2\|\theta^\varepsilon\|_s)$ for sufficiently large s by equation (98) and the bounds on R^ε and $\tilde{R}^\varepsilon$. This immediately yields

$$\left\|\left(I+\left(I+\frac{2\pi}{L^\varepsilon}\widetilde{A}^\varepsilon\Lambda\right)^{-1}T[\theta^\varepsilon]\right)^{-1}\left(I+\frac{2\pi}{L^\varepsilon}\widetilde{A}^\varepsilon\Lambda\right)^{-1}F^\varepsilon\right\|_0 \leq C_1(1+\|\gamma^\varepsilon\|_{s-1}^2)\exp(C_2\|\theta^\varepsilon\|_s),$$

by Assumption 5.4 and Lemma 5.3. In conclusion, using the inequality (83), $T[\theta^\varepsilon]$ is a bounded operator from H^0 to H^{s-2}; together with the estimates on R^ε and $\tilde{R}^\varepsilon$, we have $\|Q^\varepsilon\|_{s-2} \leq C_1(1+\|\gamma^\varepsilon\|_{s-1}^2)\exp(C_2\|\theta^\varepsilon\|_s)$.

Finally the estimate on ν^ε is readily obtained from its definition (75), the Schwartz inequality and using previous estimates.

□

5.3 Existence and the energy estimate

We now prove that solutions of our mollified system exist, by using the Picard Theorem. This gives existence on a time interval which depends badly upon the mollification parameter, ε. We then prove an energy estimate, and this allows us to extend the existence time to be uniform in ε. We begin by stating the Picard Theorem on a Banach space; there are many such statements of this theorem in the literature, including in [21].

Theorem 5.6 (Picard)**.** *Let $O \subset B$ be an open subset of a Banach space B. Let $F : O \to B$. Assume that F is locally Lipschitz continuous, i.e., that for all $x \in O$, there exists an open neighborhood of x, $U_x \subseteq O$ and $c > 0$ such that for all $x_1, x_2 \in U_x$,*

$$\|F(x_1) - F(x_2)\|_B \leq c\|x_1 - x_2\|_B.$$

Then, for every $x_0 \in O$, there exists $T > 0$ and $x \in C^1((-T, T); O)$ such that x is the solution of the initial value problem

$$\frac{dx}{dt} = F(x), \quad x(0) = x_0.$$

In order to use Picard theorem, the open set $\mathcal{O}$ is as defined previously.

Theorem 5.7. *Let $(\theta_0, \gamma_0) \in \mathcal{O}$ be given with $\langle\langle \sin(\theta_0) \rangle\rangle = 0$. There exists $T_\varepsilon > 0$ and $(\theta^\varepsilon, \gamma^\varepsilon) \in C^1((-T_\varepsilon, T_\varepsilon); \mathcal{O})$ such that $(\theta^\varepsilon, \gamma^\varepsilon)$ is the unique solution of the initial value problem given by* (73) *with initial data (θ_0, γ_0).*

We do not provide further details, but the proof that the relevant function has the correct mapping properties and is Lipschitz is now routine. We have demonstrated the existence of solutions to the mollified system. We would like to pass to the limit as $\varepsilon \to 0^+$. However, we cannot do this yet, as the time interval from Theorem 5.7 could go to zero as ε vanishes. Our next step is to prove an energy estimate, uniformly in ε, for the solutions $(\theta^\varepsilon, \gamma^\varepsilon)$. We can then use the continuation theorem for ordinary differential equations on a Banach space to find that the solutions of the mollified system exist on a common time interval; after this, we will be able to pass to the limit as the regularization parameter vanishes.

Theorem 5.8. *Let $(\theta_0, \gamma_0) \in \mathcal{O}$ be given, with $\langle\langle \sin(\theta_0) \rangle\rangle = 0$. Let $\varepsilon > 0$ be given. Let $(\theta^\varepsilon, \gamma^\varepsilon) \in C([0, T], \mathcal{O})$ be a solution of* (73)*, with initial conditions (θ_0, γ_0). Then there exists constants $c_1 \in (0, \infty)$, $c_2 \in (0, 1)$ and $c_3 \in (0, \infty)$, depending only on $s, \bar{d}_1, \bar{d}_2, \bar{d}_3, \|\theta_0\|_s$ and $\|\gamma_0\|_{s-1}$, such that*

$$\|\theta^\varepsilon\|_s^2 + \|\gamma^\varepsilon\|_{s-1}^2 \leq -c_1 \ln(c_2 - c_3 t). \tag{118}$$

Proof. As in the example of Section 2, We define an energy functional E as

$$E(t) = E_0(t) + E_1(t) + E_2(t) + E_3(t), \tag{119}$$

where the definitions of E_0, E_1, E_2 and E_3 are given by

$$E_0(t) = \frac{1}{2}\int_0^{2\pi} (\theta^\varepsilon)^2 + (\gamma^\varepsilon)^2 d\alpha, \tag{120}$$

$$E_1(t) = \frac{(L^\varepsilon)^2 \bar{A}^\varepsilon}{4\pi^2}\int_0^{2\pi} (\partial_\alpha^s \theta^\varepsilon)^2 d\alpha, \tag{121}$$

$$E_2(t) = \frac{1}{2}\int_0^{2\pi} (\partial_\alpha^{s-2}\gamma^\varepsilon)(H\partial_\alpha^{s-1}\gamma^\varepsilon) d\alpha, \tag{122}$$

$$E_3(t) = \frac{\pi \widetilde{A}^\varepsilon}{L^\varepsilon}\int_0^{2\pi} (H\partial_\alpha^{s-1}\gamma^\varepsilon)^2 d\alpha. \tag{123}$$

We see that $\bar{A}^\varepsilon \geq 0$ and $\widetilde{A}^\varepsilon \geq 0$ for all t, so the energy is well defined. (Also, recall that L^ε is bounded above and below.)

To begin, we take the time derivative of E_0:

$$\frac{dE_0}{dt} = \int_0^{2\pi} \theta^\varepsilon \theta_t^\varepsilon + \gamma^\varepsilon \gamma_t^\varepsilon d\alpha.$$

Since s is sufficiently largely, it is immediate, from the evolution equations (73), the definitions (86) and (87), and related equations, as well as the estimates from Lemma 5.5, that the following inequality holds:

$$\frac{dE_0}{dt} \leq C_1 \exp(C_2 E). \tag{124}$$

We next take the time derivative of E_1:

$$\frac{dE_1}{dt} = \frac{((L^\varepsilon)^2 \bar{A}^\varepsilon)_t}{4\pi^2}\int_0^{2\pi} (\partial_\alpha^s \theta^\varepsilon)^2 d\alpha + \frac{(L^\varepsilon)^2 \bar{A}^\varepsilon}{2\pi^2}\int_0^{2\pi} (\partial_\alpha^s \theta^\varepsilon)(\partial_\alpha^s \theta_t^\varepsilon)\, d\alpha. \tag{125}$$

We note that $\partial_\alpha \nu^\varepsilon = 0$, so there is no contribution from ν^ε in (125). To proceed with (125), we will focus on the second term on the right-side hand. Applying ∂_α^s to θ_t^ε (see (86)), we get

$$\partial_\alpha^s \theta_t^\varepsilon = \frac{2\pi^2}{(L^\varepsilon)^2}\chi_\varepsilon H(\partial_\alpha^{s+1}\gamma^\varepsilon) + \frac{2\pi}{L^\varepsilon}\chi_\varepsilon \left(V_W^\varepsilon(\chi_\varepsilon \partial_\alpha^{s+1}\theta^\varepsilon)\right) + \Phi_1, \tag{126}$$

where Φ_1 is given by the formula

$$\Phi_1 = \frac{2\pi}{L^\varepsilon}\chi_\varepsilon \sum_{j=1}^{s} \binom{s}{j} \partial_\alpha^j V_W^\varepsilon(\chi_\varepsilon \partial_\alpha^{s+1-j}\theta^\varepsilon) + \frac{2\pi}{L^\varepsilon}\partial_\alpha^s(\boldsymbol{m}^{\boldsymbol{\varepsilon}} \cdot \boldsymbol{n}^{\boldsymbol{\varepsilon}}).$$

All of the summands here involve at most s derivative of θ^ε and V_W^ε. Therefore, the estimates of Lemma 5.5 immediately imply that $\|\Phi_1\|_0 \leq C_1 \exp(C_2 E)$. We plug (126) into (125), and we use the fact that χ_ε

is self-adjoint:

$$\frac{dE_1}{dt} = \frac{((L^\varepsilon)^2\bar{A}^\varepsilon)_t}{4\pi^2}\int_0^{2\pi}(\partial_\alpha^s\theta^\varepsilon)^2\,d\alpha + \bar{A}^\varepsilon\int_0^{2\pi}(\partial_\alpha^s\theta^\varepsilon)\left(\chi_\varepsilon H\partial_\alpha^{s+1}\gamma^\varepsilon\right)\,d\alpha \\ + \frac{L^\varepsilon\bar{A}^\varepsilon}{\pi}\int_0^{2\pi}(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)V_W^\varepsilon(\chi_\varepsilon\partial_\alpha^{s+1}\theta^\varepsilon)\,d\alpha + \frac{(L^\varepsilon)^2\bar{A}^\varepsilon}{2\pi^2}\int_0^{2\pi}(\partial_\alpha^s\theta^\varepsilon)\Phi_1\,d\alpha. \quad (127)$$

For the third term on the right-hand side of (127), by integrating by parts we have

$$\left|-\frac{L^\varepsilon\bar{A}^\varepsilon}{2\pi}\int_0^{2\pi}(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)^2\partial_\alpha V_W^\varepsilon\,d\alpha\right| \le C_1\exp(C_2E).$$

Now we compute $\dfrac{dE_2}{dt}$:

$$\frac{dE_2}{dt} = \int_0^{2\pi}\left(H\partial_\alpha^{s-1}\gamma^\varepsilon\right)\left(\partial_\alpha^{s-2}\gamma_t^\varepsilon\right)\,d\alpha.$$

Next, we substitute using the formula (87):

$$\frac{dE_2}{dt} = \int_0^{2\pi}\left(H\partial_\alpha^{s-1}\gamma^\varepsilon\right)\left(-\bar{A}^\varepsilon\chi_\varepsilon\partial_\alpha^{s+2}\theta^\varepsilon - 2\widetilde{A}^\varepsilon\frac{\pi}{L^\varepsilon}H(\partial_\alpha^{s-1}\gamma_t^\varepsilon) - \partial_\alpha^{s-2}\left(\frac{4\pi^2\widetilde{A}^\varepsilon}{(L^\varepsilon)^2}\chi_\varepsilon(V_W^\varepsilon\chi_\varepsilon H(\gamma_{\alpha\alpha}^\varepsilon))\right) + \partial_\alpha^{s-2}Q^\varepsilon\right)d\alpha.$$

We treat this similarly to dE_1/dt, expanding as follows:

$$\frac{dE_2}{dt} = -\bar{A}^\varepsilon\int_0^{2\pi}\left(H\partial_\alpha^{s-1}\gamma^\varepsilon\right)\chi_\varepsilon\partial_\alpha^{s+2}\theta^\varepsilon\,d\alpha - \frac{2\widetilde{A}^\varepsilon\pi}{L^\varepsilon}\int_0^{2\pi}\left(H\partial_\alpha^{s-1}\gamma^\varepsilon\right)H\partial_\alpha^{s-1}\gamma_t^\varepsilon\,d\alpha \\ - \frac{4\pi^2\widetilde{A}^\varepsilon}{(L^\varepsilon)^2}\int_0^{2\pi}\left(\chi_\varepsilon H\partial_\alpha^{s-1}\gamma^\varepsilon\right)V_W^\varepsilon\chi_\varepsilon H\partial_\alpha^s\gamma^\varepsilon\,d\alpha \\ - \frac{4\pi^2\widetilde{A}^\varepsilon}{(L^\varepsilon)^2}\int_0^{2\pi}\left(\chi_\varepsilon H\partial_\alpha^{s-1}\gamma^\varepsilon\right)\sum_{j=1}^{s-2}\binom{s-2}{j}(\partial_\alpha^j V_W^\varepsilon)(\chi_\varepsilon H\partial_\alpha^{s-j}\gamma^\varepsilon)\,d\alpha + \int_0^{2\pi}\left(H\partial_\alpha^{s-1}\gamma^\varepsilon\right)\partial_\alpha^{s-2}Q^\varepsilon\,d\alpha. \quad (128)$$

We deal with the terms on the right-hand side of (128) one by one. We integrate by parts twice in the first term on the right-hand side of (128), and we use that χ_ε is self-adjoint:

$$-\bar{A}^\varepsilon\int_0^{2\pi}\left(H\partial_\alpha^{s-1}\gamma^\varepsilon\right)\chi_\varepsilon\partial_\alpha^{s+2}\theta^\varepsilon\,d\alpha = -\bar{A}^\varepsilon\int_0^{2\pi}\left(\chi_\varepsilon H\partial_\alpha^{s+1}\gamma^\varepsilon\right)\partial_\alpha^s\theta^\varepsilon\,d\alpha,$$

which is same as the second term on the right-hand side of (127) with the opposite sign.

Next, we rewrite the second term on the right-hand side of (128):

$$-\frac{2\widetilde{A}^\varepsilon\pi}{L^\varepsilon}\int_0^{2\pi}H\partial_\alpha^{s-1}\gamma^\varepsilon H\partial_\alpha^{s-1}\gamma_t^\varepsilon d\alpha = -\frac{\widetilde{A}^\varepsilon\pi}{L^\varepsilon}\frac{d}{dt}\int_0^{2\pi}\left(H\partial_\alpha^{s-1}\gamma^\varepsilon\right)^2\,d\alpha. \quad (129)$$

We integrate by parts in the third term on the right-hand side of (128):

$$-\frac{4\pi^2\widetilde{A}^\varepsilon}{(L^\varepsilon)^2}\int_0^{2\pi}\left(\chi_\varepsilon H\partial_\alpha^{s-1}\gamma^\varepsilon\right)V_W^\varepsilon\chi_\varepsilon H\partial_\alpha^s\gamma^\varepsilon\,d\alpha=\frac{2\pi^2\widetilde{A}^\varepsilon}{(L^\varepsilon)^2}\int_0^{2\pi}\left(\chi_\varepsilon H\partial_\alpha^{s-1}\gamma^\varepsilon\right)^2\partial_\alpha V_W^\varepsilon\,d\alpha.$$

For the fourth term on the right-hand side of (128), we see that $\sum_{j=1}^{s-2}\binom{s-2}{j}(\partial_\alpha^j V_W^\varepsilon)(\chi_\varepsilon H\partial_\alpha^{s-j}\gamma^\varepsilon)$ involves at most $s-1$ derivatives of γ^ε and $s-2$ of V_W^ε, so, using Lemma 5.5, we conclude

$$\left\|\sum_{j=1}^{s-2}\binom{s-2}{j}(\partial_\alpha^j V_W^\varepsilon)(\chi_\varepsilon H\partial_\alpha^{s-j}\gamma^\varepsilon)\right\|_0\leq C_1\exp(C_2E).$$

And for the fifth term on the right-hand side of (128), we see $\|\partial_\alpha^{s-2}Q^\varepsilon\|_0\leq C_1\exp(C_2E)$ by Lemma 5.5.

Finally we compute $\dfrac{dE_3}{dt}$:

$$\frac{dE_3}{dt}=-\frac{d}{dt}\left(\frac{\pi\widetilde{A^\varepsilon}}{L^\varepsilon}\right)\int_0^{2\pi}(H\partial_\alpha^{s-1}\gamma^\varepsilon)^2d\alpha+\frac{\pi\widetilde{A}^\varepsilon}{L^\varepsilon}\frac{d}{dt}\int_0^{2\pi}(H\partial_\alpha^{s-1}\gamma^\varepsilon)^2d\alpha. \tag{130}$$

When adding, (130) will provide a cancellation with (129). Now we add (127), (128), and (130), and we then conclude that

$$\frac{dE_1}{dt}+\frac{dE_2}{dt}+\frac{dE_3}{dt}\leq C_1\exp(C_2E) \tag{131}$$

(We note that we have not discussed explicitly the terms which contain L_t^ε, but these are immediately bounded, using the definition of the energy and Lemma 5.5.) So we combine the time derivatives of E_0, E_1, E_2, and E_3, finding

$$\frac{dE}{dt}\leq C_1\exp(C_2E),$$

which implies

$$E(t)\leq\frac{-\ln(e^{-C_2E(0)}-C_1C_2t)}{C_2}.$$

This completes the proof of the theorem. □

5.4 Passing to the limit: Existence and regularity

Now that we have both established the existence of solutions $(\theta^\varepsilon,\gamma^\varepsilon)$ and established bounds on these which are independent of ε, we are in a position to pass to the limit as ε vanishes.

Theorem 5.9. *Let $(\theta_0,\gamma_0)\in\mathcal{O}$ be given, with θ_0 satisfying $\langle\!\langle\sin(\theta_0)\rangle\!\rangle=0$. There exists $T>0$ and $(\theta,\gamma)\in C([0,T];\bar{\mathcal{O}})$ such that (θ,γ) satisfies* (20) *and* (50) *with $(\theta(\cdot,0),\gamma(\cdot,0))=(\theta_0,\gamma_0)$.*

Proof. Note that the set $\bar{\mathcal{O}}$ is the closure of the open set $\mathcal{O}$. Theorem 5.8 implies that the norm of solutions of the mollified problem (73), $(\theta^\varepsilon, \gamma^\varepsilon)$, will not blow up in a short time. Indeed, it indicates that the solutions $(\theta^\varepsilon, \gamma^\varepsilon)$ are bounded independently of ε. Furthermore, the solutions all exist on a common time interval, by the continuation theorem for autonomous ODE on a Banach space. That is, there exists $T > 0$, such that for all $\varepsilon > 0$, solutions of the initial value problem $(\theta^\varepsilon, \gamma^\varepsilon)$ are in $C([0,T]; \mathcal{O})$, where $\mathcal{O}$ is a bounded subset of $H^s \times H^{s-1}$, and since we have taken s sufficiently large ($s \geq 5$ suffices for the present purpose), this implies that each of $\theta^\varepsilon_\alpha, \theta^\varepsilon_t, \gamma^\varepsilon_\alpha$ and γ^ε_t are uniformly bounded periodic functions. Thus θ^ε and γ^ε are bounded and equicontinuous families. By the Arzela-Ascoli theorem, there exists $(\theta, \gamma) \in C([0, 2\pi] \times [0,T]) \times C([0, 2\pi] \times [0,T])$ such that a subsequence of $(\theta^\varepsilon, \gamma^\varepsilon)$ converges uniformly to (θ, γ) on $[0, 2\pi] \times [0,T]$. We will now show that this pair (θ, γ) is in the closure of $\mathcal{O}$, and also that it solves the non-mollified evolution equations.

Since each of θ and γ are in $C([0, 2\pi] \times [0,T])$, we see that they are also in $L^2([0, 2\pi])$ at each time. By Lemma 4.1, we can conclude that the subsequence of $(\theta^\varepsilon, \gamma^\varepsilon)$ actually converges to (θ, γ) in $H^{s'} \times H^{s'-1}$ for any $1 \leq s' < s$. Moreover for any such s', we have $(\theta, \gamma) \in L^\infty([0,T]; H^{s'} \times H^{s'-1})$.

Since the divided differences for the solutions θ^ε are bounded by $\bar{d}_3$, we can pass it to the limit, finding

$$|q_1[\theta](\alpha, \alpha')| \geq \bar{d}_3 > 0 \quad \forall \alpha, \alpha'.$$

Furthermore, all the solutions θ^ε satisfy

$$\langle\!\langle \sin(\theta^\varepsilon) \rangle\!\rangle = \frac{1}{2\pi} \int_0^{2\pi} \sin(\theta^\varepsilon) d\alpha = 0.$$

Since along our subsequence θ^ε converges uniformly to θ, we can pass to the limit. That is, $\mathbb{P}(\sin(\theta)) = \sin(\theta)$. We also can pass to the limit to find that ν^ε goes to 0 as ε vanishes. Recall the definition of ν^ε and take the limit

$$\lim_{\varepsilon \to 0^+} \nu^\varepsilon = -\frac{L}{4\pi^2} \int_0^{2\pi} \mathcal{B}_1 \cos(\theta) d\alpha = -\frac{1}{2\pi} \int_0^{2\pi} (U_\alpha + V\theta_\alpha) \cos(\theta) d\alpha.$$

We have previously calculated this integral; it is equal to

$$\frac{L_t}{2\pi L} \int_0^{2\pi} \sin(\theta) d\alpha = 0.$$

We now integrate (73) from $t = 0$ to any time $t \in [0,T]$,

$$(\theta^\varepsilon, \gamma^\varepsilon) = (\theta_0, \gamma_0) + \int_0^t (\mathcal{B}_1^\varepsilon + \nu^\varepsilon, \mathcal{B}_2^\varepsilon) \, d\tau.$$

We have established sufficient regularity thus far to be able to pass to the limit here (along our subsequence), finding

$$(\theta, \gamma) = (\theta_0, \gamma_0) + \int_0^t (\mathcal{B}_1, \mathcal{B}_2) \, d\tau.$$

Differentiating this with respect to time, we have shown that (θ, γ) is indeed a solution of (52), as desired.

It remains to demonstrate that $(\theta, \gamma) \in C([0,T]; H^s \times H^{s-1})$. For any $t \in [0,T]$, the sequence $(\theta^\varepsilon(\cdot,t), \gamma^\varepsilon(\cdot,t))$ is uniformly bounded with respect to both ε and t in $H^s \times H^{s-1}$. The unit ball of a Hilbert space is weakly compact, so there exists a weak limit along a subsequence. This weak limit must clearly be $(\theta(\cdot,t), \gamma(\cdot,t))$, which implies that $(\theta,\gamma) \in L^\infty([0,T]; H^s \times H^{s-1})$.

To show continuity in time in $H^s \times H^{s-1}$, we must show that for any $t_* \in [0,T]$, we have

$$\lim_{t \to t_*} \|\theta(\cdot,t) - \theta(\cdot,t_*)\|_s + \|\gamma(\cdot,t) - \gamma(\cdot,t_*)\|_{s-1} = 0.$$

Note that the above limit is a one-sided limit when $t_* = 0$ or $t_* = T$. We know that the space $H^s \times H^{s-1}$ is a Hilbert space. To establish convergence in a Hilbert space, it is sufficient to establish weak convergence, plus convergence of the norm.

Firstly, for weak convergence, we only focus on θ as an example. There is no difference with γ. Let any s' satisfying $0 < s' < s$ be given. We know that $\theta(\cdot,t) \to \theta(\cdot,t_*)$ in $H^{s'}$. Let $\phi \in H^{-s}$ be given. By Sobolev embedding, $H^{-s'}$ is dense in H^{-s} since $-s' > -s$. By Theorem 5.8, there exists a constant $K > 0$ such that for any $t \in [0,T]$, $\|\theta(\cdot,t)\|_s \leq K$. Let $\delta > 0$ be given, and choose $\phi_\delta \in H^{-s'}$ such that $\|\phi - \phi_\delta\|_{-s} \leq \dfrac{\delta}{3(1+K)}$. Then

$$\langle \theta(\cdot,t) - \theta(\cdot,t_*), \phi \rangle = \langle \theta(\cdot,t) - \theta(\cdot,t_*), \phi_\delta \rangle + \langle \theta(\cdot,t) - \theta(\cdot,t_*), \phi - \phi_\delta \rangle.$$

The second term on the right-hand side is bounded by $\dfrac{2K\delta}{3(1+K)}$, and thus by $2\delta/3$. And the first integral can be made smaller than $\dfrac{\delta}{3}$ by taking t sufficiently close to t_* since $\theta(\cdot,t)$ converges to $\theta(\cdot,t_*)$ in $H^{s'}$. This demonstrates the weak convergence. Finally, all that remains is to prove that the $H^s \times H^{s-1}$ norm of (θ,γ) is continuous in time. We omit the details of this, as the argument is identical to the corresponding argument in [3] for the vortex sheet with surface tension, or the corresponding argument in Chapter 3 of [21] for the incompressible Euler equations. □

6 Uniqueness and continuous dependence

We have proved above that solutions (θ,γ) exist in the space $H^s \times H^{s-1}$. Now we assume that we have two solutions (θ,γ) and (θ',γ') in this space and estimate the difference in a lower regularity space $H^3 \times H^2$. This space is chosen to be high enough so that the estimates have positive powers of derivatives, but low enough so that the terms can be bounded by $\|\theta\|_s$ and $\|\gamma\|_{s-1}$.

Before beginning, we note that we will use the estimates in Lemma 5.5 with unregularized quantities, i.e. for $\varepsilon = 0$. We also remark that the estimate that we now perform, for $(\theta - \theta', \gamma - \gamma')$ in $H^3 \times H^2$, will be very similar to the energy estimate above for the existence proof.

Theorem 6.1. *Let* $(\theta_0, \gamma_0) \in \mathcal{O}$ *and* $(\theta_0', \gamma_0') \in \mathcal{O}$ *be given, with* $\langle\langle \sin(\theta_0) \rangle\rangle = \langle\langle \sin(\theta_0') \rangle\rangle = 0$. *Let* $T > 0$ *be such that there exists* $(\theta,\gamma) \in C([0,T], \mathcal{O})$ *which solves* (52) *with initial value* $(\theta(\cdot,0), \gamma(\cdot,0)) = (\theta_0, \gamma_0)$, *and there exists* $(\theta', \gamma') \in C([0,T], \mathcal{O})$ *which solves* (52) *with initial value* $(\theta'(\cdot, 0), \gamma'(\cdot, 0)) = (\theta'_0, \gamma'_0)$. *There exists* $c > 0$ *such that*

$$\sup_{t\in[0,T]} (|L - L'| + \|\theta - \theta'\|_3 + \|\gamma - \gamma'\|_2) \leq c(|L(0) - L'(0)| + \|\theta_0 - \theta'_0\|_3 + \|\gamma_0 - \gamma'_0\|_2)). \tag{132}$$

Moreover, the solution of the initial value problem (52) *with initial data* $(\theta_0, \gamma_0) \in \mathcal{O}$ *is unique.*

Proof. We define

$$E_d(t) = Z_0(t) + Z_1(t) + Z_2(t) + Z_3(t), \tag{133}$$

where

$$Z_0(t) = \frac{1}{2}(L - L')^2 + \frac{1}{2}\int_0^{2\pi} (\theta - \theta')^2 + (\gamma - \gamma')^2 d\alpha, \tag{134}$$

$$Z_1(t) = \frac{L^2 \bar{A}}{4\pi^2}\int_0^{2\pi} (\partial_\alpha^3 \theta - \partial_\alpha^3 \theta')^2 d\alpha, \tag{135}$$

$$Z_2(t) = \frac{1}{2}\int_0^{2\pi} (\partial_\alpha \gamma - \partial_\alpha \gamma')(H\partial_\alpha^2 \gamma - H\partial_\alpha^2 \gamma') d\alpha, \tag{136}$$

$$Z_3(t) = \frac{\pi \widetilde{A}}{L}\int_0^{2\pi} (H\partial_\alpha^2 \gamma - H\partial_\alpha^2 \gamma')^2 d\alpha. \tag{137}$$

It is important to be able to get the Lipschitz estimates on the various quantities corresponding to Lemma 5.5 associated to $(\theta - \theta', \gamma - \gamma')$ with $s = 3$. The simplest quantities are the unit tangent and normal vectors which are bounded by standard Lipschitz estimates for sine and cosine:

$$\|\boldsymbol{t} - \boldsymbol{t}'\|_3 = \|(\cos(\theta) - \cos(\theta'), \sin(\theta) - \sin(\theta'))\|_3 \leq C\|\theta - \theta'\|_3 \leq CE_d^{1/2}. \tag{138}$$

And similarly,

$$\|\boldsymbol{n} - \boldsymbol{n}'\|_3 \leq CE_d^{1/2}.$$

Since $z_a = \dfrac{L}{2\pi}\mathcal{C}(\boldsymbol{t})$, a bound for $z_\alpha - z'_\alpha$ follows:

(139)

$$|z_\alpha - z'_\alpha\|_3 \leq \left\|\frac{L - L'}{2\pi}\mathcal{C}(\boldsymbol{t})\right\|_3 + \frac{L'}{2\pi}\|\mathcal{C}(\boldsymbol{t}) - \mathcal{C}(\boldsymbol{t}')\|_3 \leq CE_d^{1/2}. \tag{140}$$

Next we estimate $\boldsymbol{m} - \boldsymbol{m}'$. We rewrite

$$\mathcal{C}(\boldsymbol{m} - \boldsymbol{m}') = I + II,$$

where

$$I = z_\alpha K[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) - z'_\alpha K[z'_d]\left(\left(\frac{\gamma'}{z'_\alpha}\right)_\alpha\right),$$

$$II = \frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) - \frac{z'_\alpha}{2i}\left[H, \frac{1}{(z'_\alpha)^2}\right]\left(z'_\alpha\left(\frac{\gamma'}{z'_\alpha}\right)_\alpha\right).$$

We rewrite I by adding and subtracting:

$$I = (z_\alpha - z'_\alpha)K[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) + z'_\alpha(K[z_d] - K[z'_d])\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) + z'_\alpha K[z'_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha - \left(\frac{\gamma'}{z'_\alpha}\right)_\alpha\right).$$

With our uniform bounds on θ and γ, and using Lemma 4.2 and Lemma 4.4, the first term on the right-hand side is bounded in H^3 by $CE_d^{1/2}$. For the second term, we apply Lemma 4.5, and we thus see that this term is also bounded in H^3 by $CE_d^{1/2}$. For the third term on the right-hand side, its norm in H^3 is bounded by $CE_d^{1/2}$ since $K[z'_d]$ is a smoothing operator by Lemma 4.4.

We now consider the term II. We start by adding and subtracting:

$$II = \frac{z_\alpha - z'_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) + \frac{z'_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2} - \frac{1}{(z'_\alpha)^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)$$
$$+ \frac{z'_\alpha}{2i}\left[H, \frac{1}{(z'_\alpha)^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha - z'_\alpha\left(\frac{\gamma'}{z'_\alpha}\right)_\alpha\right).$$

The first term on the right-hand side can be immediately bounded in H^3 by $CE_d^{1/2}$. The third term can be bounded in H^3 by $CE_d^{1/2}$ by Lemma 4.6. For the second term on the right-hand side, we bound it not by using smoothing properties of commutators, but by not regarding it as a commutator at all. That is, we write the term out:

$$\frac{z'_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2} - \frac{1}{(z'_\alpha)^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) = \frac{z'_\alpha}{2i}H\left(\left(\frac{1}{z_\alpha^2}-\frac{1}{(z'_\alpha)^2}\right)z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)+\frac{z'_\alpha}{2i}\left(\frac{1}{z_\alpha^2}-\frac{1}{(z'_\alpha)^2}\right)H\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right).$$

Each of the terms on the right-hand side can clearly be bounded in H^3 by $CE_d^{1/2}$. This completes the estimate of $\boldsymbol{m} - \boldsymbol{m}'$. We conclude

$$\|\boldsymbol{m} - \boldsymbol{m}'\|_3 \le CE_d^{1/2}. \tag{141}$$

Now we estimate $\|\boldsymbol{W} - \boldsymbol{W}'\|_2$. We first expand this, as we have several times before:

$$\mathcal{C}^*(\boldsymbol{W}) - \mathcal{C}^*(\boldsymbol{W}') = \left(K[z_d] - K[z'_d]\right)\gamma + \frac{1}{2i}H(\gamma/z_\alpha - \gamma'/z_\alpha) + K[z'_d](\gamma - \gamma') + \frac{1}{2i}H(\gamma'/z_\alpha - \gamma'/z'_\alpha). \tag{142}$$

By Lemma 4.4 and Lemma 4.5, $\|\boldsymbol{W} - \boldsymbol{W}'\|_2$ is thus bounded by $CE_d^{1/2}$. This immediately yields the fact that $\|U - U'\|_2$ is also bounded by $CE_d^{1/2}$ since $U = \boldsymbol{W} \cdot \boldsymbol{n}$. From formula (16) and the Schwartz inequality, we immediately find the following estimate:

$$|L_t - L'_t| \le CE_d^{1/2}. \tag{143}$$

The bound on $\|V - V'\|_3$ follows from equation (17) and the bound on $\|U - U'\|_2$. After adding and subtracting several times, and by the above estimates on $\boldsymbol{t} - \boldsymbol{t}'$ and $\boldsymbol{m} - \boldsymbol{m}'$, and by equation (37), the following estimate can be found:

$$\|V_W - V_W'\|_3 \leq CE_d^{1/2}.$$

Now we can conclude that

$$\|\theta_t - \theta_t'\|_0 \leq CE_d^{1/2}. \tag{144}$$

For the estimate of $\dfrac{dZ_0}{dt}$, we still must estimate $\|\gamma_t - \gamma_t'\|_0$. Recalling (47), by adding and subtracting several times, we have the following estimate:

$$\|\tilde{R} - \tilde{R}'\|_1 \leq CE_d^{1/2}.$$

We now pay attention to $R - R'$. First, we find $\|z_t - z_t'\|_2 \leq CE_d^{1/2}$ since $z_t = \mathcal{C}(U\boldsymbol{n} + V\boldsymbol{t})$, and making use of the above Lipschitz estimates for $U, V, \boldsymbol{t}, \boldsymbol{n}$. There are many Hilbert commutators and terms involving $K[z_d]$ in the definition of R, and we only show the estimate for one such term in $\|R_3 - R_3'\|_2$ as an example. In the term on which we focus, we use the smoothing properties of Hilbert commutators carefully; for other terms, the similar properties of $K[z_d]$ are used. For the term we select, we add and subtract:

$$\left[H, \frac{1}{z_\alpha}\right]\left(z_\alpha\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)_\alpha\right) - \left[H, \frac{1}{z_\alpha'}\right]\left(z_\alpha'\left(\frac{\gamma' z_{t\alpha}'}{z_\alpha'}\right)_\alpha\right) = \left[H, \frac{1}{z_\alpha} - \frac{1}{z_\alpha'}\right]\left(z_\alpha\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)_\alpha\right)$$
$$+ \left[H, \frac{1}{z_\alpha'}\right]\left((z_\alpha - z_\alpha')\left(\frac{\gamma z_{t\alpha}'}{z_\alpha'}\right)_\alpha\right) + \left[H, \frac{1}{z_\alpha'}\right]\left(z_\alpha'\left(\frac{(\gamma - \gamma')z_{t\alpha}'}{z_\alpha'}\right)_\alpha\right) + \left[H, \frac{1}{z_\alpha'}\right]\left(z_\alpha'\left(\gamma'(\frac{z_{t\alpha}}{z_\alpha} - \frac{z_{t\alpha}'}{z_\alpha'})\right)_\alpha\right).$$

The first term and the second term on the right-hand side are bounded in H^2 by $CE_d^{1/2}$ since $\|z_\alpha - z_\alpha'\|_2$ is bounded by $CE_d^{1/2}$ and $z_\alpha(\frac{\gamma z_{t\alpha}}{z_\alpha})_\alpha$ is bounded in H^{s-3}. The third term on the right-hand side is bounded in H^2 by $CE_d^{1/2}$ since by Lemma 4.6,

$$\left\|\left[H, \frac{1}{z_\alpha'}\right]\left(z_\alpha'\left(\frac{(\gamma - \gamma')z_{t\alpha}'}{z_\alpha'}\right)_\alpha\right)\right\|_2 \leq C\left\|\frac{1}{z_\alpha'}\right\|_2\left\|\left(z_\alpha'\left(\frac{(\gamma - \gamma')z_{t\alpha}'}{z_\alpha'}\right)_\alpha\right)\right\|_0.$$

The fourth term on the right-hand side is bounded in H^2 by $CE_d^{1/2}$, since by Lemma 4.6,

$$\left\|\left[H, \frac{1}{z_\alpha'}\right]\left(z_\alpha'\left(\gamma'(\frac{z_{t\alpha}}{z_\alpha} - \frac{z_{t\alpha}'}{z_\alpha'})\right)_\alpha\right)\right\|_2 \leq \left\|\frac{1}{z_\alpha'}\right\|_2\left\|\left(z_\alpha'\left(\gamma'(\frac{z_{t\alpha}}{z_\alpha} - \frac{z_{t\alpha}'}{z_\alpha'})\right)_\alpha\right)\right\|_0.$$

We omit the remaining details for our estimate of $R - R'$ since they are similar to the above. We now make the conclusion that

$$\|R - R'\|_2 \leq CE_d^{1/2}.$$

Recalling the formula for F (95), we see that this involves four derivatives of θ and two derivatives of γ; we estimate thusly:

$$\left\|\left(I + \frac{2\pi}{L}\widetilde{A}\Lambda\right)^{-1}(F - F')\right\|_0 \leq CE_d^{1/2}. \tag{145}$$

We must understand the regularity of γ_t. First, using the evolution equation (94), we write γ_t as

$$\gamma_t = \left(I + \left(I + \frac{2\pi}{L}\widetilde{A}\Lambda\right)^{-1} T[\theta]\right)^{-1} \left(I + \frac{2\pi}{L}\widetilde{A}\Lambda\right)^{-1} F.$$

Since $\theta \in H^s, \gamma \in H^{s-1}, \tilde{R} \in H^{s-2}$ and $R \in H^{s-1}$, we have $F \in H^{s-4}$. We see from Lemma 5.3 that $\gamma_t \in H^0$ when $F \in H^0$ for sufficiently large s. Next, using the estimate (83), we can demonstrate that $T[\theta]\gamma_t \in H^{s-2}$. Since $\gamma_t = \left(I + \frac{2\pi}{L}\widetilde{A}\Lambda\right)^{-1} (-T[\theta]\gamma_t + F)$ by the evolution equation (94), we see that $\gamma_t \in H^{s-3}$.

Now we focus on the Lipschitz estimate for $T[\theta]$. Since we can decompose $T[\theta]$ as commutators and terms involving $K[z_d]$, we can claim that for any $f \in H^2$,

$$\|T[\theta]f - T[\theta']f\|_1 \leq CE_d^{1/2}\|f\|_2. \tag{146}$$

The proof of (146) is analogous to the proof of Lemma 5.2 by adding and subtracting several times, so we only give one interesting term as an example:

$$\begin{aligned}\left\|K[z_d]\left(\left(\frac{f}{z_\alpha}\right)_\alpha\right) - K[z_d']\left(\left(\frac{f}{z_\alpha'}\right)_\alpha\right)\right\|_1 &\leq \left\|K[z_d]\left(\left(\frac{f}{z_\alpha}\right)_\alpha\right) - K[z_d']\left(\left(\frac{f}{z_\alpha}\right)_\alpha\right)\right\|_1 \\ &\quad + \left\|K[z_d']\left(\left(\frac{f}{z_\alpha}\right)_\alpha - \left(\frac{f}{z_\alpha'}\right)_\alpha\right)\right\|_1 \\ &\leq C\|\theta - \theta'\|_1 \left\|\left(\frac{f}{z_\alpha}\right)_\alpha\right\|_1 + C\|z_d\|_3 \left\|\left(\frac{f}{z_\alpha}\right)_\alpha - \left(\frac{f}{z_\alpha'}\right)_\alpha\right\|_0 \leq CE_d^{1/2}\|f\|_2.\end{aligned}$$

Since $\gamma_t \in H^2$ for sufficiently large s, the estimate on $(T[\theta] - T[\theta'])\gamma_t$ follows:

$$\|(T[\theta] - T[\theta'])\gamma_t\|_1 \leq CE_d^{1/2}. \tag{147}$$

We are ready to estimate $\|\gamma_t - \gamma_t'\|_0$. Using the evolution equation (94), we add and subtract to get the following equation:

$$\gamma_t - \gamma_t' = -\frac{2\widetilde{A}\pi}{L}H(\gamma_{\alpha t} - \gamma_{\alpha t}') - \left(\frac{2\widetilde{A}\pi}{L} - \frac{2\widetilde{A}'\pi}{L'}\right)H(\gamma_{\alpha t}') - T[\theta](\gamma_t - \gamma_t') - (T[\theta] - T[\theta'])\gamma_t' + F - F'. \tag{148}$$

By inequalities (145) and (147) the following estimate can be found:

$$\|F_d\|_0 := \left\|\left(I + \frac{2\pi}{L}\widetilde{A}\Lambda\right)^{-1}\left(-(\frac{2\widetilde{A}\pi}{L} - \frac{2\widetilde{A}'\pi}{L'})H(\gamma_{\alpha t}') - (T[\theta] - T[\theta'])\gamma_t' + F - F'\right)\right\|_0 \leq CE_d^{1/2}.$$

By Lemma 5.3, we finally have

$$\|\gamma_t - \gamma_t'\|_0 = \left\|\left(I + \left(I + \frac{2\pi}{L}\widetilde{A}\Lambda\right)^{-1} T[\theta]\right)^{-1} F_d\right\|_0 \leq CE_d^{1/2}. \tag{149}$$

Furthermore, we make a conclusion that

$$\frac{dZ_0}{dt} \leq CE_d. \tag{150}$$

We still must estimate the growth of Z_1, Z_2, and Z_3; first, we work with $Q - Q'$. Since we have $Q = -T[\theta]\gamma_t + \tilde{R} + R$, we have the following for $Q - Q'$:

$$Q - Q' = -T[\theta](\gamma_t - \gamma_t') - (T[\theta] - T[\theta'])(\gamma_t') + \tilde{R} - \tilde{R}' + R - R'. \tag{151}$$

By the estimate of $\|\gamma_t - \gamma_t'\|_0$ and the smoothing estimate of $T[\theta]$ (83), and the previous Lipschitz estimates for $T[\theta]$, $\tilde{R}$, and R, we can conclude that

$$\|Q - Q'\|_1 \leq CE_d^{1/2}. \tag{152}$$

Now, we will take the time derivative of Z_1 :

$$\frac{dZ_1}{dt} = \frac{(L^2\bar{A})_t}{4\pi^2}\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')^2 d\alpha + \frac{L^2\bar{A}}{2\pi^2}\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')(\partial_\alpha^3\theta_t - \partial_\alpha^3\theta_t')d\alpha.$$

We apply ∂_α^3 to equation (20), finding

$$\partial_\alpha^3\theta_t = \frac{2\pi^2}{L^2}H\partial_\alpha^4\gamma + \frac{2\pi}{L}V_W\partial_\alpha^4\theta + Y_1,$$

where Y_1 is defined as

$$Y_1 = \frac{2\pi}{L}\partial_\alpha^3 V_W\partial_\alpha\theta + \frac{6\pi}{L}\partial_\alpha^2 V_W\partial_\alpha^2\theta + \frac{6\pi}{L}\partial_\alpha V_W\partial_\alpha^3\theta + \frac{2\pi}{L}\partial_\alpha^3(\boldsymbol{m}\cdot\boldsymbol{n}).$$

Substituting yields the following:

$$\begin{aligned}\frac{dZ_1}{dt} &= \frac{(L^2\bar{A})_t}{4\pi^2}\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')^2 d\alpha + \frac{L^2\bar{A}}{2\pi^2}\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')\left(\frac{2\pi^2}{L^2}H\partial_\alpha^4\gamma - \frac{2\pi^2}{(L')^2}H\partial_\alpha^4\gamma'\right)d\alpha \\ &\quad + \frac{L^2\bar{A}}{2\pi^2}\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')\left(\frac{2\pi}{L}V_W\partial_\alpha^4\theta - \frac{2\pi}{L'}V_W'\partial_\alpha^4\theta'\right)d\alpha + \frac{L^2\bar{A}}{2\pi^2}\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')(Y_1 - Y_1')d\alpha.\end{aligned} \tag{153}$$

There are four integrals on the right-hand side of (153). We estimate them one by one. The first term on the right-hand side of (153) is obviously bounded by CE_d. We will add and subtract to rewrite the other three terms. The second term is equal to

$$\bar{A}\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')(H\partial_\alpha^4\gamma - H\partial_\alpha^4\gamma')d\alpha + \frac{L^2\bar{A}}{2\pi^2}\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')\left(\frac{2\pi^2}{L^2} - \frac{2\pi^2}{(L')^2}\right)H\partial_\alpha^4\gamma' d\alpha. \tag{154}$$

The first term in (154) will cancel with another term later. The second term in (154) is bounded by CE_d since L is bounded (above and below) and $\|\gamma'\|_4$ is bounded if $s \geq 5$. We now work with the third term of equation (153); after adding and subtracting, it is

$$\frac{L^2 \bar{A}}{2\pi^2} \int_0^{2\pi} (\partial_\alpha^3 \theta - \partial_\alpha^3 \theta') \frac{2\pi}{L} V_W (\partial_\alpha^4 \theta - \partial_\alpha^4 \theta') d\alpha + \frac{L^2 \bar{A}}{2\pi^2} \int_0^{2\pi} (\partial_\alpha^3 \theta - \partial_\alpha^3 \theta') \left(\frac{2\pi}{L} V_W - \frac{2\pi}{L'} V_W' \right) \partial_\alpha^4 \theta' d\alpha. \tag{155}$$

The first term in (155) is bounded by CE_d since we can integrate by parts. The second term in (155) is bounded by CE_d since

$$\frac{2\pi}{L} V_W - \frac{2\pi}{L'} V_W' = \left(\frac{2\pi}{L} - \frac{2\pi}{L'} \right) V_W' + \frac{2\pi}{L'} (V_W - V_W')$$

and $\|V_W - V_W'\|_3 \leq CE_d^{1/2}$. The last term of equation (153) requires an estimate for $\|Y_1 - Y_1'\|_0$. After adding and subtracting, it is bounded by $CE_d^{1/2}$ because it only involves at most 3 derivatives of $\theta - \theta'$ and at most 2 derivatives of $\gamma - \gamma'$.

We now take the time derivative of Z_2 :

$$\frac{dZ_2}{dt} = \int_0^{2\pi} (\partial_\alpha \gamma_t - \partial_\alpha \gamma_t')(H\partial_\alpha^2 \gamma - H\partial_\alpha^2 \gamma') d\alpha.$$

Using (50), we have the following equation:

$$\gamma_{\alpha t} = -\bar{A} \partial_\alpha^5 \theta - \frac{2\widetilde{A}\pi}{L} H(\partial_\alpha^2 \gamma_t) - \frac{4\widetilde{A}\pi^2}{L^2} (V_W)_\alpha H(\gamma_{\alpha\alpha}) - \frac{4\widetilde{A}\pi^2}{L^2} V_W H(\partial_\alpha^3 \gamma) + Q_\alpha.$$

Notice that $\|Q_\alpha - Q_\alpha'\|_0 \leq CE_d^{1/2}$ and $-\frac{4\widetilde{A}\pi^2}{L^2}(\partial_\alpha V_W) H(\gamma_{\alpha\alpha})$ contains at most a second derivative of γ. That is,

$$\left\| -\frac{4\widetilde{A}\pi^2}{L^2} (V_W)_\alpha H(\gamma_{\alpha\alpha}) + \frac{4\widetilde{A}'\pi^2}{(L')^2} (V_W')_\alpha H(\gamma_{\alpha\alpha}') \right\|_0 \leq CE_d^{1/2}.$$

Now we are going to deal with

$$-\frac{2\widetilde{A}\pi}{L} H(\partial_\alpha^2 \gamma_t) + \frac{2\widetilde{A}'\pi}{L'} H(\partial_\alpha^2 \gamma_t') = -\left(\frac{2\widetilde{A}\pi}{L} - \frac{2\widetilde{A}'\pi}{L'} \right) H(\partial_\alpha^2 \gamma_t') - \frac{2\widetilde{A}\pi}{L} H(\partial_\alpha^2 \gamma_t - \partial_\alpha^2 \gamma_t').$$

The first term of the above equation is bounded by $CE_d^{1/2}$ because $\gamma_t \in H^2$ for sufficiently large s.

Summing up the Z_2 calculation, we have

$$\frac{dZ_2}{dt} \leq -\bar{A} \int_0^{2\pi} (\partial_\alpha^5 \theta - \partial_\alpha^5 \theta')(H\partial_\alpha^2 \gamma - H\partial_\alpha^2 \gamma') d\alpha - \frac{2\widetilde{A}\pi}{L} \int_0^{2\pi} H(\partial_\alpha^2 \gamma_t - \partial_\alpha^2 \gamma_t')(H\partial_\alpha^2 \gamma - H\partial_\alpha^2 \gamma') d\alpha$$
$$- \int_0^{2\pi} \left(\frac{4\widetilde{A}\pi^2}{L^2} V_W H\partial_\alpha^3 \gamma - \frac{4\widetilde{A}'\pi^2}{(L')^2} V_W' H\partial_\alpha^3 \gamma' \right) (H\partial_\alpha^2 \gamma - H\partial_\alpha^2 \gamma') d\alpha + CE_d. \tag{156}$$

Adding $\frac{dZ_1}{dt}$, $\frac{dZ_2}{dt}$, and $\frac{dZ_3}{dt}$, we can cancel the first two terms on the right-hand side equation (156). Furthermore, by adding, subtracting and integrating by parts, the third term of the above right-hand side is bounded by CE_d. So we now can conclude that

$$\frac{dE_d}{dt} \leq CE_d. \tag{157}$$

Solving the inequality, we find

$$E_d(t) \leq E_d(0)e^{Ct}.$$

This implies both continuous dependence and uniqueness. This also completes the proof. □

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