NON-EXISTENCE OF SMALL-AMPLITUDE DOUBLY PERIODIC WAVES FOR DISPERSIVE EQUATIONS

DAVID M. AMBROSE AND J. DOUGLAS WRIGHT

ABSTRACT. We formulate the question of existence of spatially periodic, time-periodic solutions for evolution equations as a fixed point problem, for certain temporal periods. We prove that if a certain estimate applies for the Duhamel integral, then time-periodic solutions cannot be arbitrarily small. This provides a partial analogue in the spatially periodic case of scattering results for dispersive equations on the real line, as scattering implies the non-existence of small-amplitude traveling waves. Furthermore, it also complements small-divisor methods (e.g, the Craig-Wayne-Bourgain method) for proving the *existence* of small-amplitude time-periodic solutions (again, for frequencies in certain set).

Non-existence d'onde de petites amplitudes doublement périodiques pour les équations dispersives
Résumé: Nous exprimons le problème d'existence de solutions périodiques en temps et en espace d'opérateurs d'évolution sous forme de problèmes de points fixes, pour certaines périodes de temps. Nous prouvons que si une certaine estimation pour l'integrale de Duhamel existe, alors les solutions périodiques en temps ne peuvent etre arbitrairement petites. Cela donne des résultats analogues pour le cas de la diffusion d'ondes périodiques dans l'espace sur la droite réelle, puisque la diffusion implique la non-existence d'onde de petites amplitudes. De plus, nos résultats viennent compléter les méthodes des petits diviseurs (comme par exemple la méthode de Craig-Wayne-Bourgain) pour prouver l'existence de solutions périodiques en temps de petites amplitudes (pour des frequences dans un certain ensemble).

1. Introduction

In this paper, we introduce a framework for proving the non-existence of time-periodic, spatially periodic solutions of dispersive partial differential equations. Our framework starts from the point of view of time-periodic solutions as fixed points of the Poincaré map. We decompose the Poincaré map, using a linear solution operator and a nonlinear mapping, and factoring out the linear part. We are then able to demonstrate nonexistence of time-periodic solutions if certain estimates are satisfied.

Since we have factored out a linear mapping, we must have a bound on the inverse of this linear mapping. We prove such an estimate by small divisor techniques. The result is that the relevant operator can be bounded as $|k|^p$, where k is the Fourier variable, for any $p > 1$, for certain possible frequencies. Since the linear estimate gives a bound as $|k|^p$, there must be a compensating gain in regularity from the nonlinear solution operator; this gain of regularity comes from dispersive smoothing estimates.

The framework we develop here is complementary to two significant lines of work in the area of dispersive partial differential equations. These are (a) scattering results for dispersive partial differential equations on the real line, and (b) existence results for time-periodic, spatially periodic waves *via* Nash-Moser-type small divisor methods.

There are many results in the literature on scattering for small-amplitude solutions of dispersive partial differential equations in free space, and we cannot attempt to discuss them all. We mention that a few such results are [23], [6], [14], [22], [19]. We will mention in detail the results of [23]. There, Strauss shows for two dispersive equations, a nonlinear Schrödinger equation and a generalized Korteweg-de Vries (KdV) equation, that all sufficiently small solutions decay in time. This implies the non-existence of small-amplitude coherent structures, such as traveling waves. These results, however, use in a fundamental way that the domain is unbounded, and it can be difficult to see an analogue on a periodic interval. The present results, however, do provide a partial analogue; as will be described in detail below, we demonstrate a mechanism by which dispersion can prevent the existence of small-amplitude coherent structures, in certain cases, on the periodic interval.

There are two primary types of proof in the literature for the existence of time-periodic, spatially periodic waves. For equations which happen to be completely integrable, such as the KdV equation or the Benjamin-Ono equation, it is often the case that explicit formulas for time-periodic solutions can be written down [9], [20], [1], [2], [25]. For more general equations, however, integrable systems techniques are not available, and small divisor methods of Nash-Moser type, especially as developed by Craig and Wayne, and further developed by Bourgain, are commonly used [24], [7], [4], [5]. This type of method has been further developed for applications to water waves by Plotnikov, Toland, and Iooss [21], [15], and for perturbations of the Benjamin-Ono equation by Baldi [3].

The contrast between the results of these two methods is stark; for completely integrable equations, continua of time-periodic solutions are found. As shown explicitly for the Benjamin-Ono equation in [1], [2], [25], nontrivially time-periodic solutions bifurcate from traveling waves (which are trivially time-periodic), and a continuous family of such waves eventually terminates at a different traveling wave. For such a continuous family, the temporal period of these waves varies continuously, taking values in a closed interval of positive diameter. For the small-divisor results, however, the estimates only close when certain frequencies are discarded; thus, the typical result is that time-periodic solutions exist when the frequency is chosen from a set of positive measure (a Cantor set). At the frequencies in this Cantor set, the result is that small-amplitude time-periodic solutions exist; typical equations treated in this manner include nonlinear wave equations and nonlinear Schrödinger equations.

We note that these Nash-Moser-type results are silent on the question of whether better results are possible: while they indicate that time-periodic solutions do exist with the frequency chosen from a Cantor set, they do not indicate that time-periodic solutions do not exist for any of the remaining frequencies. In fact, in the cases for which we have complete information (the completely integrable equations), we know that time-periodic solutions do exist at a continuum of frequencies. The results of the present paper demonstrate how one might show that the existence results of small-divisor type may in fact be nearly optimal, for certain equations – the main result (Corollary 3 below) shows that for certain equations, dispersive smoothing estimates imply the nonexistence of small-amplitude, time-periodic solutions for certain frequencies. Thus, we demonstrate an avenue by which the question of existence and non-existence of small-amplitude time-periodic solutions can be nearly fully understood, for certain dispersive partial differential equations. We mention that a similar result was proved by de la Llave for nonlinear wave equations [8].

In Section 2, we explain our formulation as a fixed-point problem, and we prove a general theorem. In Section 3, we prove an estimate for a linear evolution, and we then find our main result as a corollary. We conclude in Section 4 with a discussion of the applicability of the main result to specific dispersive equations.

2. Problem formulation and a general theorem

In this section, we formulate the time-periodic solutions problem for a nonlinear equation, and for the time being, we proceed abstractly. Consider the evolution equation

$$u_t = Au + Nu, \tag{1}$$

where A is a linear operator and N is a nonlinear operator. Let S_L be the solution operator for the related equation

$$v_t = Av; \tag{2}$$

that is, for any $t \in \mathbb{R}$, we can write $S_L(t)v_0 = e^{At}v_0$. Let $S(t)$ be the solution operator for the equation (1) (starting from time zero and continuing until time t), and let $S_D(t)$ be the difference of S and S_L : $S_D(t) = S(t) - S_L(t)$. We can understand S_D better by using the usual Duhamel formula:

$$S(t)u_0 = u(\cdot, t) = S_L(t)u_0 + \int_0^t S_L(t-\tau)N(u(\tau))\ d\tau.$$

Subtracting $S_L(t)u_0$ from both sides, we see the following formula for S_D :

$$S_D(t)u_0 = \int_0^t S_L(t-\tau)N(u(\tau))\, d\tau. \tag{3}$$

A time-periodic solution of (1) with temporal period T corresponds to a fixed point of the Poincaré map: if u_0 satisfies

$$u_0 = S(T)u_0, \tag{4}$$

then u_0 is the initial data for a time-periodic solution. We rewrite (4) by putting everything on one side of the equation, and also by using the decomposition $S = S_L + S_D$:

$$\left[(I - S_L(T)) - S_D(T)\right] u_0 = 0. \tag{5}$$

Here, I is the identity operator. We may continue to rewrite this, if T is a period for which (2) lacks time-periodic solutions. Indeed, for such a T, the linear operator $I - S_L(T)$ has trivial kernel, and thus is invertible. We let W be the set of such values:

$$W = \left\{t \in (0,\infty) : \ker(I - S_L(t)) = \{0\}\right\}.$$

For $T \in W$, we then factor $I - S_L(T)$ out of (5):

$$(I - S_L(T))[I - (I - S_L(T))^{-1}S_D(T)]u_0 = 0. \tag{6}$$

We give the name $K(T)$ to the final operator on the left-hand side:

$$K(T) = (I - S_L(T))^{-1}S_D(T).$$

We then see that a time-periodic solution of (1) with temporal period $T \in W$ corresponds to solutions of the equation

$$(I - K(T))u_0 = 0. \tag{7}$$

If there is a function space, $\mathbf{X}$, and a constant $r_0 > 0$, such that for all $x \in \mathbf{X}$ satisfying $0 < \|x\|_{\mathbf{X}} \le r_0$, we have $\|K(T)x\|_{\mathbf{X}} < \|x\|_{\mathbf{X}}$, then clearly (7) has no nontrivial solutions of size r_0 or smaller. We have proven the following:

Theorem 1. *Let $T \in W$ be given. Let $\mathbf{X}$ be a Banach space, such that $K(T) : \mathbf{X} \to \mathbf{X}$. Assume there exists $r_0 > 0$ such that for all $x \in \mathbf{X}$ satisfying $0 < \|x\|_{\mathbf{X}} \le r_0$, $\|K(T)x\|_{\mathbf{X}} < \|x\|_{\mathbf{X}}$. Then, for any nontrivial time-periodic solution, u, of (1) with period T,*

$$\inf_{t\in[0,T]} \|u(\cdot,t)\|_{\mathbf{X}} > r_0.$$

For a given $T \in W$, if we knew that $(I - S_L(T))^{-1}$ were a bounded linear operator on some function space $\mathbf{X}$, then in order to be able to apply Theorem 1, it would be sufficient to find a constant $q > 1$ such that for sufficiently small x, we have $\|S_D(T)x\|_{\mathbf{X}} \le c\|x\|^q_{\mathbf{X}}$. This is not what we will show, however, since we are unable to demonstrate that $(1 - S_L(T))^{-1}$ is a bounded operator, from a Sobolev space to itself, for the evolution equations in which we are interested. To be precise, in Section 3, we will demonstrate a bound for the inverse operator which shows that the symbol of the inverse operator acts like $|k|^p$ for some $p > 1$, where k is the variable in Fourier space; thus, the inverse acts like differentiation of order p. In order to compensate for this, we will need to use smoothing properties of $S_D(T)$. If we know that $(I - S_L(T))^{-1} : \mathbf{X} \to \mathbf{Y}$ is a bounded operator, and $S_D(T) : \mathbf{Y} \to \mathbf{X}$ is a bounded operator, with $\|S_D(T)u_0\|_{\mathbf{X}} \le c\|u_0\|^q_{\mathbf{Y}}$, for some $q > 1$, then this will be sufficient to apply Theorem 1.

3. The inverse of the linear operator

We now assume that the linearized evolution equation (linearized about zero) is, after taking the Fourier transform, $\mathcal{F}$,

$$\mathcal{F}w_t(k,t) = ik^r \mathcal{F}w(k,t), \tag{8}$$

for some constant, $r > 0$. Using this, we make an estimate of small-divisor-type for the symbol of the inverse of our operator.

Theorem 2. *For any $p > 1$, there exists a set $W_p \subseteq W$ of positive measure such that for all k,*

$$\left|\mathcal{F}\left[(1 - S_L(T))^{-1}\right](k)\right| \leq c|k|^p.$$

Sketch of Proof: Fix $p > 1$. We let ω be defined through the equation $T = \frac{2\pi}{\omega}$. We consider $\omega \in [1,2]$, which is to say, we consider $T \in [\pi, 2\pi]$. Fix k, and define $Q[\omega]$ as

$$Q[\omega] = \left|\left(1 - e^{ik^r T}\right)^{-1}\right| = \left|\left(1 - e^{2\pi i k^r/\omega}\right)^{-1}\right|.$$

In order for $Q[\omega]$ to even be defined, we must not divide by zero; notice that we divide by zero here if and only if there exists $n \in \mathbb{N}$ such that ω equals k^r/n.

Let $n_* \in \mathbb{N}$ be given such that $\omega_* = \frac{k^r}{n_*} \in [1,2]$. We consider $\omega = \omega_* \pm \varepsilon$, for small ε. With this choice of ω, we have $Q[\omega] = \left|1 - \cos\left(\frac{2\pi k^r}{\frac{k^r}{n_*} \pm \varepsilon}\right) - i \sin\left(\frac{2\pi k^r}{\frac{k^r}{n_*} \pm \varepsilon}\right)\right|^{-1}$. We make the following computation:

$$\frac{2\pi k^r}{\frac{k^r}{n_*} \pm \varepsilon} = \frac{2\pi n_*}{1 \pm \frac{\varepsilon n_*}{k^r}} \approx 2\pi n_* \mp \frac{2\pi\varepsilon n_*^2}{k^r}.$$

We then have the following:

$$\cos\left(\frac{2\pi k^r}{\frac{k^r}{n_*} \pm \varepsilon}\right) \approx \cos\left(2\pi n_* \mp \frac{2\pi\varepsilon n_*^2}{k^r}\right) = \cos\left(\frac{2\pi\varepsilon n_*^2}{k^r}\right) \approx 1 - \frac{1}{2}\left(\frac{2\pi\varepsilon n_*^2}{k^r}\right)^2,$$

$$\sin\left(\frac{2\pi k^r}{\frac{k^r}{n_*} \pm \varepsilon}\right) \approx \sin\left(2\pi n_* \mp \frac{2\pi\varepsilon n_*^2}{k^r}\right) = \mp\sin\left(\frac{2\pi\varepsilon n_*^2}{k^r}\right) \approx \mp\frac{2\pi\varepsilon n_*^2}{k^r}.$$

This implies $Q[\omega] \approx \dfrac{|k|^r}{2\pi\varepsilon n_*^2}$.

As we said above, this approximation for $Q[\omega]$ is valid as long as ε is small, and in particular, we need $\frac{\varepsilon n_*^2}{|k|^r} \ll 1$. Using our $p > 1$ and letting $c_0 > 0$ be constant, we choose $\varepsilon = c_0|k|^{-(r+p)}$. Since n_* is commensurate to k^r, we see that this leaves our estimate for $Q[\omega]$ as $Q[\omega] \leq c|k|^p$, for some constant $c > 0$, as long as $\omega \notin [\omega_* - \varepsilon, \omega_* + \varepsilon]$, for any valid choice of n_*.

Now, there are $O(k^r)$ valid choices for n_*, so we have removed $O(k^r)$ intervals of width ε from the set of possible values of ω; with our choice of ε, this means we have removed a set with measure proportional to $|k|^{-p}$, for our fixed value of k. Summing over all values of k, we see that we have removed only a set of finite measure, since $p > 1$ (this is the reason that we chose $p > 1$ in the beginning of the proof). By simply choosing the constant c_0 to be sufficiently small, we end up with the set W_p having positive measure. ∎

We can then state our main result, which follows from Theorem 1 and Theorem 2. This says that a gain of regularity on the Duhamel integral implies the non-existence of small-amplitude time-periodic solutions. The gain of regularity on the Duhamel integral will be discussed in the final section.

Corollary 3. *Let S_D be as above, assuming that the linear evolution operator is given by (8). Assume there exist $s \in \mathbb{R}$, $p > 1$, $q > 1$, and $c > 0$ such that for all T, $S_D(T) : H^s \to H^{s+p}$, with the estimate*

$$\|S_D(T)u_0\|_{H^{s+p}} \leq c\|u_0\|_{H^s}^q$$

for all sufficiently small $u_0 \in H^s$. Then, any nontrivial time-periodic solutions of (1) with temporal period $T \in W_p$ cannot be arbitrarily small.

4. Discussion

Our main result, Corollary 3, shows that small-amplitude doubly periodic waves cannot exist if certain estimates are satisfied for the Duhamel integral. This raises the question, then, of whether the corresponding estimates for the Duhamel integral hold. The required estimate on the Duhamel integral is a smoothing estimate; since the inverse of the linear operator loses p derivatives for $p > 1$, we must have a gain of p derivatives in the Duhamel term. There are results in the literature that are similar to what is required in the present case. Specifically, for the generalized KdV equation (posed on the real line) and the KdV equation (posed on a periodic interval), there are smoothing results for the Duhamel integral [18], [11].

The result of Linares and Scialom [18] demonstrates a gain of one derivative for the Duhamel integral, as compared to the initial data. The result of Erdogan and Tzirakis [11], similarly, is that the Duhamel integral gains $1 - \varepsilon$ derivatives, as compared to the initial data (for any $\varepsilon > 0$). In general, for a dispersive equation with dispersion relation of order r (as in (8) above), one expects a gain of $(r-1)/2$ derivatives from the effect of dispersion [17]. For equations of KdV-type, one has $r = 3$, and thus one expects a gain of one derivative. It certainly seems no accident that the smoothing on the Duhamel integral demonstrated in [18] and [11] is on this same order.

Given that Linares and Scialom and Erdogan and Tzirakis have shown a gain like one derivative for the Duhamel integral in KdV-like equations, it seems natural to expect that with stronger dispersion, one will find a greater gain of regularity. For simplicity, the authors are currently focusing on the following equation, which has fifth-order dispersion:

$$u_t + \partial_x^5 u = u u_x. \tag{9}$$

The necessary smoothing result for the operator S_D associated to (9) holds, and the authors' proof of the following theorem will appear in a subsequent work:

Theorem 4. *For sufficiently large s, there exists $T > 0$ such that the initial value problem (9) with $u(\cdot, 0) = u_0 \in H^s$ has a unique classical solution $u \in C([0,T]; H^s)$. Given $u_0 \in H^s$, let $u_1(\cdot, t) = S_L(t)u_0$, for all $t \in [0,T]$. Let $u_2(\cdot,t) = u(\cdot,t) - u_1(\cdot,t) = S_D(t)u_0$, for all $t \in [0,T]$. Then $u_2 \in C([0,T]; H^{s+2})$.*

For any interval $[T_1, T_2] \subseteq (0, \infty)$, there exists $\delta > 0$ and $c > 0$ such that if $u_0 \in H^s$ satisfies $\|u_0\|_{H^s} \leq \delta$, then the above solution is in $C([0,T_2]; H^s)$, with the estimate $\|S_D(T)u_0\|_{H^{s+2}} \leq c\|u_0\|_{H^s}^2$ for all $T \in [T_1, T_2]$.

The existence theory and the final estimate are established by the energy method. The proof of smoothing follows the lines of the Erdogan-Tzirakis argument, but is simpler as there is a benefit to the stronger dispersion; in particular, the use of Bourgain spaces is avoided. Given this theorem, in light of Corollary 3, we have the following:

Corollary 5. *Fix $p \in (1, 2]$, and let W_p and s be as above. There exists $r_0 > 0$ such that if u is a non-constant time-periodic solution of (9) with temporal period $T \in W_p$, then $\inf_{t \in [0,T]} \|u(\cdot, t)\|_{H^s} > r_0$.*

Finally, we remark that it seems likely that the corresponding result would hold for other equations with greater-than-third-order dispersion. Specifically, one such equation is the Kawahara equation, $u_t + \partial_x^5 u - \partial_x^3 u + uu_x = 0$, which has been rigorously justified as a model for water waves with surface tension [10]. Also of interest are fourth-order Schrödinger equations, such as $i\psi_t + \Delta\psi + |\psi|^{2\sigma}\psi + \varepsilon\Delta^2\psi = 0$, for $\sigma > 0$ and $\varepsilon > 0$, which can arise by including higher-order corrections when deriving Schrödinger equations from Maxwell's equations [16], [12], [13]. As the referee has remarked to the authors, Theorem 2 holds in one spatial dimension, so the corresponding results for Schrödinger equations should hold in the case of one spatial dimension.

Acknowledgments

DMA gratefully acknowledges support from the National Science Foundation through grants DMS-1008387 and DMS-1016267. JDW gratefully acknowledges support from the National Science Foundation through grants DMS-0908299 and DMS-1105635. The authors thank Jean-Luc Bouchot for providing the translation of the abstract and title. The authors thank the anonymous referee for helpful suggestions.

References

[1] D.M. Ambrose and J. Wilkening. Global paths of time-periodic solutions of the Benjamin-Ono equation connecting pairs of traveling waves. *Commun. Appl. Math. Comput. Sci.*, 4:177–215, 2009.

[2] D.M. Ambrose and J. Wilkening. Computation of time-periodic solutions of the Benjamin-Ono equation. *J. Nonlinear Sci.*, 20(3):277–308, 2010.

[3] P. Baldi. Periodic solutions of fully nonlinear autonomous equations of Benjamin–Ono type. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30(1):33–77, 2013.

[4] J. Bourgain. Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE. *Internat. Math. Res. Notices*, 1994(11):475ff., approx. 21 pp. (electronic), 1994.

[5] J. Bourgain. Construction of periodic solutions of nonlinear wave equations in higher dimension. *Geom. Funct. Anal.*, 5(4):629–639, 1995.

[6] F.M. Christ and M.I. Weinstein. Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. *J. Funct. Anal.*, 100(1):87–109, 1991.

[7] W. Craig and C.E. Wayne. Newton's method and periodic solutions of nonlinear wave equations. *Comm. Pure Appl. Math.*, 46(11):1409–1498, 1993.

[8] R. de la Llave. Variational methods for quasi-periodic solutions of partial differential equations. In *Hamiltonian systems and celestial mechanics (Pátzcuaro, 1998)*, volume 6 of *World Sci. Monogr. Ser. Math.*, pages 214–228. World Sci. Publ., River Edge, NJ, 2000.

[9] B.A. Dubrovin. A periodic problem for the Korteweg-de Vries equation in a class of short-range potentials. *Funkcional. Anal. i Priložen.*, 9(3):41–51, 1975.

[10] W.-P. Düll. Validity of the Korteweg-de Vries approximation for the two-dimensional water wave problem in the arc length formulation. *Comm. Pure Appl. Math.*, 65(3):381–429, 2012.

[11] M.B. Erdoğan and N. Tzirakis. Global smoothing for the periodic KdV evolution. *Int. Math. Res. Not. IMRN*, 2013(20):4589–4614, 2013.

[12] G. Fibich, B. Ilan, and G. Papanicolaou. Self-focusing with fourth-order dispersion. *SIAM J. Appl. Math.*, 62(4):1437–1462 (electronic), 2002.

[13] G. Fibich, B. Ilan, and S. Schochet. Critical exponents and collapse of nonlinear Schrödinger equations with anisotropic fourth-order dispersion. *Nonlinearity*, 16(5):1809–1821, 2003.

[14] J. Ginibre and G. Velo. Scattering theory in the energy space for a class of nonlinear Schrödinger equations. In *Semigroups, theory and applications, Vol. I (Trieste, 1984)*, volume 141 of *Pitman Res. Notes Math. Ser.*, pages 110–120. Longman Sci. Tech., Harlow, 1986.

[15] G. Iooss, P.I. Plotnikov, and J.F. Toland. Standing waves on an infinitely deep perfect fluid under gravity. *Arch. Ration. Mech. Anal.*, 177(3):367–478, 2005.

[16] V.I. Karpman and A.G. Shagalov. Stability of solitons described by nonlinear Schrödinger-type equations with higher-order dispersion. *Phys. D*, 144(1-2):194–210, 2000.

[17] C.E. Kenig, G. Ponce, and L. Vega. Oscillatory integrals and regularity of dispersive equations. *Indiana Univ. Math. J.*, 40(1):33–69, 1991.

[18] F. Linares and M. Scialom. On the smoothing properties of solutions to the modified Korteweg-de Vries equation. *J. Differential Equations*, 106(1):141–154, 1993.

[19] Y. Liu. Decay and scattering of small solutions of a generalized Boussinesq equation. *J. Funct. Anal.*, 147(1):51–68, 1997.

[20] Y. Matsuno. New representations of multiperiodic and multisoliton solutions for a class of nonlocal soliton equations. *J. Phys. Soc. Japan*, 73(12):3285–3293, 2004.

[21] P.I. Plotnikov and J.F. Toland. Nash-Moser theory for standing water waves. *Arch. Ration. Mech. Anal.*, 159(1):1–83, 2001.

[22] G. Ponce and L. Vega. Nonlinear small data scattering for the generalized Korteweg-de Vries equation. *J. Funct. Anal.*, 90(2):445–457, 1990.

[23] W.A. Strauss. Dispersion of low-energy waves for two conservative equations. *Arch. Rational Mech. Anal.*, 55:86–92, 1974.

[24] C.E. Wayne. Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. *Comm. Math. Phys.*, 127(3):479–528, 1990.

[25] J. Wilkening. An infinite branching hierarchy of time-periodic solutions of the Benjamin-Ono equation. 2008. arXiv:0811.4209.

Department of Mathematics, Drexel University, 3141 Chestnut St., Philadelphia, PA 19104

Department of Mathematics, Drexel University, 3141 Chestnut St., Philadelphia, PA 19104