**Related Rates Problems**

To illustrate the general features of these problems we begin with a simple example.

A stone is dropped into a pool of water. A circular ripple spreads out in the pool. The radius increases at a rate of 3 in/sec. How fast is the area of the circle increasing?

**General Features:** Two rates of change involved. One known, one to be determined. Need a relationship between these rates of change so we can plug in the one we know and solve for the one we don't know.

**Procedure** (illustrated on the example above):

1. Draw a "snapshot" at some typical instant \( t \) and introduce names for the variables whose rates of change (derivatives) are involved.

\[ A = A(t) = \text{area of the circle at time } t \]
\[ r = r(t) = \text{radius of the circle at time } t \]
2. Write out explicitly what you know and what you're supposed to find.

**Known:** \( \frac{dr}{dt} = 3 \text{ in/sec} \)

**Find:** \( \frac{dA}{dt} \)

3. Find a relationship between the variables themselves that is true at all times \( t \).

\[ A = \pi r^2 \]

4. Differentiate this relationship with respect to \( t \).

\[ \frac{dA}{dt} = \pi \left( 2r \frac{dr}{dt} \right) = 2\pi r \frac{dr}{dt} \]

5. Plug in the derivative you know.

\[ \frac{dA}{dt} = 2\pi r (3) = 6\pi r \text{ in}^2/\text{sec} \]

**Note:** In this example, even though \( r \) increases at a constant rate, \( A \) does not, i.e., \( \frac{dA}{dt} \) depends on \( r \). This makes sense, right? The larger the circle the greater the effect on the area of a 3 in change in the radius.
MORE EXAMPLES:

1. HOW FAST DOES THE WATER LEVEL RISE IN A CYLINDRICAL CAN OF RADIUS 2 FT IF WATER IS BEING POURED IN AT A RATE OF 3 FT$^3$/SEC?

\[ V = V(t) \]
\[ = \text{VOLUME OF WATER AT TIME } t \]
\[ y = y(t) = \text{DEPTH OF WATER AT TIME } t \]

KNOWN: \[ \frac{dV}{dt} = 3 \]

FIND: \[ \frac{dy}{dt} \]

RELATIONSHIP BETWEEN \( V(t) \) AND \( y(t) \) (FORMULA FOR THE VOLUME OF A CYLINDER IS \( V = \pi r^2 h \)):

\[ V = \pi (2^2) y \]
\[ V = 4\pi y \]

DIFFERENTIATE WITH RESPECT TO \( t \):

\[ \frac{dV}{dt} = 4\pi \frac{dy}{dt} \]

SO \[ \frac{dy}{dt} = \frac{\frac{dV}{dt}}{4\pi} = \frac{3}{4\pi} \text{ FT/SEC} \]
4. GAS IS ESCAPING FROM A SPHERICAL BALLOON AT A RATE OF 2 FT³/min. HOW FAST IS THE RADIUS DECREASING WHEN IT (THE RADIUS) IS 12 FT?

\[ r = r(t) = \text{radius at time } t \]

\[ V = V(t) = \text{volume at time } t \]

KNOWN: \( \frac{dV}{dt} = -2 \) (NOTE THE MINUS SIGN)

FIND: \( \frac{dr}{dt} \) WHEN \( r = 12 \)

RELATIONSHIP BETWEEN \( V(t) \) AND \( r(t) \) (FORMULA FOR THE VOLUME OF A SPHERE):

\[ V = \frac{4}{3} \pi r^3 \]

DIFFERENTIATE WITH RESPECT TO \( t \):

\[ \frac{dV}{dt} = \frac{4}{3} \pi (3r^2 \frac{dr}{dt}) = 4\pi r^2 \frac{dr}{dt} \]

SO

\[ \frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2} = \frac{-2}{4\pi (12^2)} = -\frac{1}{288\pi} \text{ FT/min} \]
3. A ladder 26 ft long rests on the (horizontal) ground and leans against a (vertical) wall. The base of the ladder is pulled away from the wall at a rate of 4 ft/sec. How fast is the top of the ladder sliding down the wall when the base is 10 ft from the wall?

\[ y = y(t) \]

\[ x = x(t) \]

Known: \( \frac{dx}{dt} = 4 \)

Find: \( \frac{dy}{dt} \) when \( x = 10 \)

Relationship between \( x(t) \) and \( y(t) \) (Pythagorean Theorem):

\[ x^2 + y^2 = 26^2 \]

Differentiate with respect to \( t \):

\[ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \]

So

\[ \frac{dy}{dt} = - \frac{x}{y} \frac{dx}{dt} \]

Note: When \( x = 10 \), \( x^2 + y^2 = 26^2 \) gives \( 10^2 + y^2 = 26^2 \)

So \( y = 24 \).

\[ \frac{dy}{dt} = - \frac{10}{24} \frac{1}{y} = - \frac{5}{3} \text{ ft/sec} \]
4. A BALLOON IS RISING STRAIGHT UP FROM A LEVEL FIELD AND IS BEING TRACKED BY A CAMERA 500 FT FROM THE POINT OF LIFT OFF. AT THE INSTANT WHEN THE CAMERA'S ANGLE OF ELEVATION IS $\frac{\pi}{4}$, THAT ANGLE IS INCREASING AT A RATE OF 0.14 RADIANS/MIN. HOW FAST IS THE BALLOON RISING AT THAT INSTANT?

\[ y = y(t) \]

\[ \theta = \theta(t) \]

\[ \frac{d\theta}{dt} = 0.14 \text{ when } \theta = \frac{\pi}{4} \]

**KNOWN:**

**FIND:**

\[ \frac{dy}{dt} \text{ when } \theta = \frac{\pi}{4} \]

**RELATIONSHIP BETWEEN $\theta(t)$ AND $y(t)$ (DEFINITION OF $\tan \theta$)**

\[ \tan \theta = \frac{y}{500} \Rightarrow \]

\[ y = 500 \tan \theta \]

**DIFFERENTIATE WITH RESPECT TO $t$:**

\[ \frac{dy}{dt} = 500 \sec^2 \theta \frac{d\theta}{dt} \]

\[ = 500 (2) (0.14) \]

\[ = 140 \text{ FT/MIN} \]
5. Water is running out of a conical funnel at a rate of 1 in$^3$/sec. If the altitude of the funnel is 8 in and the radius of its base is 4 in, how fast is the water level dropping when it is 2 in from the top?

\[ V = V(t) \]
\[ x = x(t) \]
\[ y = y(t) \]

**Note:** It may not be obvious at this point that we need \( x = x(t) \) (radius of the water level at time \( t \)). Since \( \frac{dV}{dt} \) is given and \( \frac{dy}{dt} \) is requested, the reason is in the formula for the volume of a cone:

\[ V = \frac{1}{3} \pi r^2 h \]

which requires both the altitude and the radius.
\[ \frac{dv}{dt} = \frac{\pi}{12} (3y^2 \frac{dy}{dt}) = \frac{\pi}{4} y^2 \frac{dy}{dt} \]

so

\[ \frac{dy}{dt} = \frac{\frac{dv}{dx}}{\frac{\pi}{4} y^2} = -\frac{1}{\frac{\pi}{4} (x^2)} = -\frac{1}{\pi} \text{ m/sec} \]

6. A MAN 6 FT TALL IS WALKING AT 3 FT/SEC TOWARD A STREETLIGHT 18 FT HIGH. HOW FAST IS THE LENGTH OF HIS SHADOW CHANGING?

\[ l = l(t) = \text{LENGTH OF SHADOW AT TIME } t \]

\[ x = x(t) \]

**SIMILAR TRIANGLES** \[ \frac{l}{6} = \frac{x}{18} \Rightarrow l = \frac{1}{3} x \]

**KNOWN**: \[ \frac{dx}{dt} = -3 \]

**FIND**: \[ \frac{dl}{dt} \]

**RELATIONSHIP BETWEEN \( l(t) \) AND \( x(t) \)**: \[ l = \frac{1}{3} x \]

**DIFFERENTIATE WITH RESPECT TO \( t \)**:

\[ \frac{dl}{dt} = \frac{1}{3} \frac{dx}{dt} = \frac{1}{3} (-3) = -1 \text{ FT/SEC} \]