SOME SPECIAL VALUES YOU MUST KNOW:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\pi$</th>
<th>$\frac{3\pi}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin x$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

SOME IDENTITIES YOU MUST KNOW:

\[
\cos^2 x + \sin^2 x = 1
\]

\[
1 + \tan^2 x = \sec^2 x
\]

\[
\sin 2x = 2 \sin x \cos x
\]

\[
\cos 2x = \cos^2 x - \sin^2 x
\]

\[
\cos^2 x = \frac{1}{2} (1 + \cos 2x)
\]

\[
\sin^2 x = \frac{1}{2} (1 - \cos 2x)
\]

HERE'S ANOTHER (SEEMINGLY ODD, BUT VERY IMPORTANT) GEOMETRICAL WAY OF DEFINING FUNCTIONS:

Suppose \( f(x) \geq 0 \) on some interval \( I \)

and \( f \) is some \( a \) in \( I \).
FOR ANY $x$ IN $I$ DEFINE

$$F(x) = \begin{cases} 
\text{AREA UNDER } y = f(x) \text{ FROM } a \text{ TO } x, \text{ if } x > a \\
-\text{AREA UNDER } y = f(x) \text{ FROM } x \text{ TO } a, \text{ if } x < a
\end{cases}$$

E.G., IF $y = f(x) = \frac{1}{2} x$ ON $I = [0, \infty)$ WITH $a = 0$

$$y = f(x) = \frac{1}{2} x$$

$$F(x) = \frac{1}{2} x \left( \frac{1}{2} x \right) = \frac{1}{4} x^2$$

SPECIAL CASE: WE WILL SEE LATER THAT THE "PROPER" DEFINITION OF THE NATURAL LOGARITHM FUNCTION $\ln x$, FOR $x > 0$, IS AS THE "AREA FUNCTION" FOR $f(x) = \frac{1}{x}$ ON $(0, \infty)$ WITH $a = 1$:

$$\text{AREA} = \ln x$$
All of the "usual" properties of the natural logarithm (which you must know) can be proved from this definition:

\[ y = \ln x \]

\[ (1,0) \]

\[ \ln (1) = 0 \]
\[ \ln (ab) = \ln a + \ln b \]
\[ \ln \left( \frac{a}{b} \right) = \ln a - \ln b \]
\[ \ln \left( a^r \right) = r \ln a \]

4. Applications

**Example:** A soup company wants to manufacture a tin can in the shape of a right circular cylinder \( \text{\includegraphics[width=0.1\textwidth]{can.png}} \) that will hold 500 cm\(^3\) of soup. Express the amount of tin required to build the can as a function of the radius \( r \) of the can (and think about the problem of finding a choice of \( r \) that will minimize the amount of material required).
Surface area: \[ A = \pi r^2 + \pi r h \]
\[ A = 2\pi r^2 + 2\pi rh \]

Volume: \[ 500 = \pi r^2 h \Rightarrow h = \frac{500}{\pi r^2} \]

\[ A(r) = 2\pi r^2 + 2\pi r \left( \frac{500}{\pi r^2} \right) \]
\[ = 2\pi r^2 + \frac{1000}{r}, \quad r > 0 \]

The graph of \( A(r) \) looks roughly like this.

This value of \( r \) would require the least amount of tin. We will find it later.

5. Inverses

A function \( f \) with domain \( D \) is said to be one-to-one on \( D \) if its graph passes the "Horizontal Line Test".

[Diagrams showing one-to-one and not one-to-one functions with labels]
Find the domain of the function.

As a function of \( \theta \), express the chord length \( L \).

Consider a semicircle with radius 10 cm.

Another application/word problem:

1.1 (cont'd)

Tuesday, September 23, 2008
Domain: \( 0 < \theta < \pi \)

\[
\begin{align*}
\frac{L}{\theta} &= \frac{20 \sin \theta}{\frac{\pi}{4}} \quad \text{and} \quad L(\theta) &= 20 \sin \left( \frac{\theta}{2} \right) \\
\text{Solve for } L \\
\frac{0.2}{L} &= \frac{0.1}{\frac{\pi}{4}} = \frac{2}{\theta} \sin \theta
\end{align*}
\]
Chapter 1.3: New functions from old

Arithmetic Operations

Given functions $f$ and $g$, we can define new functions $f + g$, $f - g$, $fg$, $f/g$ as follows:

- **Addition** $f + g$:
  \[
  (f + g)(x) = f(x) + g(x)
  \]
  - **Definition**
  - **Domain**: As above, but also exclude any $x$ that makes $g(x) = 0$

- **Subtraction** $f - g$:
  \[
  (f - g)(x) = f(x) - g(x)
  \]
  - **Domain**: As above, but also exclude any $x$ that makes $g(x) = 0$

- **Multiplication** $fg$:
  \[
  (fg)(x) = f(x)g(x)
  \]
  - **Definition**
  - **Domain**: Intersection of domains of $f$ and $g$

- **Division** $f/g$:
  \[
  (f/g)(x) = \frac{f(x)}{g(x)}
  \]
  - **Definition**
  - **Domain**: Intersection of domains of $f$ and $g$, excluding any $x$ that makes $g(x) = 0$
\[
\begin{align*}
\text{Domain of } f &: \quad \frac{g}{f} \\
\text{Domain of } f + g &: \quad \frac{1}{1 + x^2} = \frac{x^2}{1 + x^2} = \frac{g(x)}{f(x)} = \frac{f(x) + g(x)}{f(x)} = \frac{x}{1 + x^2} \\
\text{Definition of } f + g &: \quad \frac{g}{f} \\
\text{Example: Consider } f(x) &= \frac{x}{1 + x^2} \\
\text{Find domain of } f + g \text{ and } \frac{g}{f}
\end{align*}
\]
Define \( f \circ g \): 

\[
(f \circ g)(x) = f(g(x))
\]

Illustration:

Composition of functions $(f \circ g)(x)$

Output of one function is the input to another.
Example: Consider $f(x) = x^2$, $g(x) = x+1$

Define $fog$ and $gof$

Define $fog$:

$$(fog)(x) = f(g(x)) = f(x+1) = (x+1)^2$$

Define $gof$:

$$(gof)(x) = g(f(x)) = g(x^2) = x^2 + 1$$
\[ (\sqrt{x})^2 + 3 = x + 3 \]

Define fog: \( f(g(x)) = f(\sqrt{x}) = f(\sqrt{x}) \)

Define fog and find its domain

Example: Consider \( f(x) = x^2 + 3 \), \( g(x) = \sqrt{x} \)

Such that \( g(x) \) is in the domain of \( f \):

All \( x \) in the domain of \( g \): The domain of fog is:
Domain of fog: \( x > 2 \)

\[
g(f(x)) = f_x \text{ is in } (L-00, +00)\]

All \( x \) in \( x > 2 \) such that

Domain of fog:

Domain of \( f \): \((L-00, +00)\)

Domain of \( g \): \( x > 0 \)

Correct method:

\[
g(f(x)) = x^2 + 3\]

E.g. \( f(g(-5)) = f(6-5) = f(1) = 3 \)

which is undefined.

The domain of fog is not all reals.

Cancel! Don't look at \((fog)(x) = x^2 + 3\) to find
Solution: Online

Define fog and find its domain

(2) \( f \circ g(x) = \sqrt{1-x}, \quad g(x) = \sqrt{x^2-1} \)

Solution: Textbook pg 29 example 3b

Define gof and find its domain

(1) \( f \circ g(x) = \sqrt{x^2+3}, \quad g(x) = \sqrt{x} \)

Practice Problems
Basic Graph Transformations

Textbook pgs 31-34

Table 1.3.2, 1.3.3, 1.3.4