Monday, September 29, 2008

Chapter 2.1: Limits (Intuitive Approach)

Definitions of Asymptotes:

(1) If any of the following are true:
\[ \lim_{x \to a^-} f(x) = +\infty, \quad \lim_{x \to a^+} f(x) = -\infty \]
\[ \lim_{x \to a^+} f(x) = +\infty, \quad \lim_{x \to a^-} f(x) = -\infty \]

then the line \( x = a \) is a **vertical asymptote of** \( y = f(x) \).
(2) If either of the following are true:
\[ \lim_{x \to +\infty} f(x) = L, \quad \lim_{x \to -\infty} f(x) = L \]
then the line \( y = L \) is a horizontal asymptote of \( y = f(x) \).

So a function can have at most two horizontal asymptotes.
Examples

(a) $\phi(y)$ undefined

(b) $\lim_{x \to y^-} \phi(x) = +\infty$

(c) $\lim_{x \to y^+} \phi(x) = +\infty$

(d) $\lim_{x \to y} \phi(x) = +\infty$

(e) $\lim_{x \to -\infty} \phi(x) = 0$

(f) $\lim_{x \to +\infty} \phi(x) = -1$

(g) Vertical Asymptotes: $x = y$
    (from (b), (c), or (d))

(h) Horizontal Asymptotes:
    \[
    \begin{align*}
    y &= 0 & \text{ (from (e), (f))} \\
    y &= -1
    \end{align*}
    \]
(a) \( f(0) = -2 \)

(b) \( \lim_{{x \to 0^-}} f(x) = 1 \)

(c) \( \lim_{{x \to 0^+}} f(x) = -\infty \)

(d) \( \lim_{{x \to 0}} f(x) \) **DNE**

(e) \( \lim_{{x \to -\infty}} f(x) = +\infty \)

(f) \( \lim_{{x \to +\infty}} f(x) = +\infty \)

(g) Vertical Asymptotes: \( x = 0 \)  
   (from (c))

(h) Horizontal Asymptotes: None  
   (from (e) and (f))
Chapter 2.2: Computing Limits

Based on the following graphs, we can see that (unsurprisingly):

\[ \lim_{x \to a} k = k \quad \text{for any constant } k \]

\[ \lim_{x \to a} x = a \]
2.2.2 Theorem. Let $a$ be a real number, and suppose that

$$
\lim_{x \to a} f(x) = L_1 \quad \text{and} \quad \lim_{x \to a} g(x) = L_2
$$

That is, the limits exist and have values $L_1$ and $L_2$, respectively. Then:

(a) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$

(b) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L_1 - L_2$

(c) $\lim_{x \to a} [f(x)g(x)] = \left(\lim_{x \to a} f(x)\right) \left(\lim_{x \to a} g(x)\right) = L_1 L_2$

(d) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L_1}{L_2}$, provided $L_2 \neq 0$

(e) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L_1}$, provided $L_1 > 0$ if $n$ is even.

Moreover, these statements are also true for the one-sided limits as $x \to a^-$ or as $x \to a^+$. 
Informally, the previous two slides mean we can compute $\lim_{x \to a} f(x)$ by simply “plugging in” and computing $f(a)$.

e.g.

1) $\lim_{x \to 3} x^2 = 3^2 = 9$

2) $\lim_{x \to -1} (x^2 + 2x) = (-1)^2 + 2(-1) = 1 - 2 = -1$
(3) \[ \lim_{x \to 5} \sqrt{x + 4} = \sqrt{5 + 4} = \sqrt{9} = 3 \]

(4) \[ \lim_{x \to 3} \frac{x^2 - 2x}{x + 1} \to \frac{3^2 - 2(3)}{3 + 1} = \frac{9 - 6}{4} = \frac{3}{4} \]

(5) \[ \lim_{x \to 1} \frac{x - 1}{2x + 1} \to \frac{1 - 1}{2(1) + 1} = \frac{0}{3} = 0 \]