Chapter 1.5 (Revisited): Review of Inverse Trig Functions

Recall: A function $f$ does not have an inverse if its graph fails the Horizontal Line Test.

So $f(x) = \sin x$ on $(-\infty, +\infty)$ has no inverse.
\( f(x) = \sin x \) with a restricted domain \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\) has an inverse

\[ f^{-1}(x) = \sin^{-1}x = \arcsin x \]

**Domain of** \( \sin^{-1}x = \) **Range of** \( \sin x : [-1, 1] \) 

**Range of** \( \sin^{-1}x = \) **Domain of** (restricted) \( \sin x : [-\frac{\pi}{2}, \frac{\pi}{2}] \)
Conceptually, $\sin^{-1} x$ is the angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is $x$.

E.g.

$$\sin^{-1} \left( \frac{1}{2} \right) = \frac{\pi}{6} \text{ because } \sin \frac{\pi}{6} = \frac{1}{2}$$

$$\sin^{-1} \left( -\frac{1}{2} \right) = -\frac{\pi}{6} \text{ because } \sin \left( -\frac{\pi}{6} \right) = -\frac{1}{2}$$
$f(x) = \cos x$ on $(-\infty, +\infty)$ has no inverse

$f(x) = \cos x$ on $(-\infty, +\infty)$ fails HLT

$f(x) = \cos x$ on $[0, \pi]$ has an inverse

$f^{-1}(x) = \cos^{-1} x = \arccos x$

$f(x) = \cos x$ on $[0, \pi]$ passes HLT

$y = \cos x$  
$0 \leq x \leq \pi$
Domain of $\cos^{-1}x = \text{Range of } \cos x : [-1, 1]$

Range of $\cos^{-1}x = \text{Domain of } \cos x : [0, \pi]$

Conceptually, $\cos^{-1}x$ represents the angle in $[0, \pi]$ whose cosine is $x$.

E.g.,

$\cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$ because $\cos \frac{\pi}{3} = \frac{1}{2}$

$\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$ because $\cos \frac{2\pi}{3} = -\frac{1}{2}$
\( f(x) = \tan x \) on its natural domain
\[ x \neq \frac{\pi}{2} + k\pi \ (k \text{ is an integer}) \] has no inverse.
$f(x) = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

has an inverse

$f^{-1}(x) = \tan^{-1} x = \arctan x$

Domain of $\tan^{-1} x = \text{Range of } \tan x: (-\infty, +\infty)$

Range of $\tan^{-1} x = \text{Domain of } \tan x: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
Conceptually, $\tan^{-1}x$ represents the angle in $(-\frac{\pi}{2}, \frac{\pi}{2})$ whose tangent is $x$.

e.g.

\[ \tan^{-1} 1 = \frac{\pi}{4} \quad \text{because} \quad \tan \frac{\pi}{4} = 1 \]

\[ \tan^{-1} (-1) = -\frac{\pi}{4} \quad \text{because} \quad \tan (-\frac{\pi}{4}) = -1 \]
Example: Find $\sec \left( \sin^{-1} \left( \frac{5}{13} \right) \right)$

Let $\theta = \sin^{-1} \left( \frac{5}{13} \right)$

So $\sin \theta = \frac{5}{13}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

\[ \text{Find } \sec \theta = \frac{1}{\cos \theta} = \frac{1}{\frac{12}{13}} = \frac{13}{12} \]