2.5.1 DEFINITION. A function $f$ is said to be *continuous at $x = c$* provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$. 

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Intuitively, a function is continuous at $x = c$ if its graph does not "break" at $x = c$. 
Some examples of f not being continuous at \( x = c \). We say \( f \) is \underline{discontinuous} at \( x = c \).

(a): \( f(c) \) is undefined

(b), (c): \( \lim_{x \to c} f(x) \) \underline{ ONE }

(d): \( \lim_{x \to c} f(x) \neq f(c) \)
Note: If \( x = c \) is an endpoint of an interval, we replace the requirement

\[
\lim_{x \to c} f(x) = f(c)
\]

to the appropriate one-sided limit, either

\[
\lim_{x \to c^-} f(x) = f(c) \quad \text{OR} \quad \lim_{x \to c^+} f(x) = f(c)
\]

This means \( f(x) \) is continuous from the left at \( x = c \).

This means \( f(x) \) is continuous from the right at \( x = c \).
e.g. \( f(x) = \sqrt{x} \) has a domain of \((0, +\infty)\) and \( \lim_{x \to 0^+} f(x) = f(0) \).

So we still say \( f(x) = \sqrt{x} \) is continuous at \( x = 0 \).
Continuity on intervals

Open intervals

$f$ is continuous on $(a, b)$ if $f$ is continuous at every value in $(a, b)$.
Closed interval

$f$ is continuous on $[a,b]$ if

1. $f$ is continuous on $(a,b)$
2. $f$ is continuous from the right at $x=a$, i.e. $\lim_{{x \to a^+}} f(x) = f(a)$
3. $f$ is continuous from the left at $x=b$, i.e. $\lim_{{x \to b^-}} f(x) = f(b)$
Example: Is $f$ continuous on the following intervals? If not, explain why.

(a) $[1,3]$: No
   $f$ is not continuous at $x=2$
   because $\lim_{x\to2} f(x) \neq f(2)$

(b) $(1,3)$: No, same reason as (a)

(c) $[1,2]$: No, $f$ is not continuous from the left at $x=2$
   because $\lim_{x\to2^-} f(x) \neq f(2)$

(d) $(1,2)$: Yes

(e) $(2,3)$: Yes

(f) $[2,3]$: No, $f$ is not continuous from the right at $x=2$
   because $\lim_{x\to2^+} f(x) \neq f(2)$
A function is said to be continuous if it is continuous at \( x = a \) for every \( a \) in its domain.

**Examples:**

- **Polynomials** are continuous (everywhere).
- **Rational functions** are continuous (where their denominators are not zero).
- Sums of continuous functions are continuous.
- Products of continuous functions are continuous.
- Quotients of continuous functions are continuous (where their denominators are not zero).
- Compositions of continuous functions are continuous.
Examples

Find the values of $x$, if any, at which $f$ is not continuous.

(1) $f(x) = \frac{x+2}{x^2-4}$

(2) $g(x) = \left| \frac{(x+3)(x^2-2)}{x^2+4} \right|$

(3) $h(x) = \left| 4 - \frac{8}{x^4+x} \right|$
(1) \[ f(x) = \frac{x+2}{x^2-4} \]

Rational function. So \( f \) is continuous everywhere except when \( x^2-4 = 0 \), i.e. when \( x = -2 \), \( x = 2 \).

So \( f \) is not continuous at \( x = -2 \), \( x = 2 \).

Review: Do \( \lim_{x \to -2} f(x) \) and \( \lim_{x \to 2} f(x) \) exist?
\( g(x) = \frac{(x+3)(x^3-2)}{x^2+4} \)

\((x+3), (x^3-2), (x^2+4)\) are polynomials and \((x^2+4) > 0\) so \(\frac{(x+3)(x^3-2)}{x^2+4}\) is continuous everywhere.

\(|x|\) is continuous everywhere.

\(f(x) = |x|, \text{ no "breaks"}\)

So \(g\) is continuous everywhere.
\[(3) \ h(x) = \left| 4 - \frac{8}{x^4 + x} \right| \]

\(h\) is continuous everywhere except when
\[x^4 + x = 0\]
\[x(x^3 + 1) = 0\]

\(x = 0 \quad x^3 + 1 = 0\)
\[x^3 = -1\]
\[x = -1\]

So \(h\) is not continuous at \(x = 0, x = -1\)
Application of Continuity:

The Intermediate Value Theorem (IVT)

Special Case: If $f$ is continuous on $[a, b]$, and $f(a)$, $f(b)$ have opposite signs, then there is at least one value $x_0$ in $(a, b)$ such that $f(x_0) = 0$. 

![Graph showing the intermediate value theorem](graph.png)
Example: Prove that $x^3 + 3x = 2$ has a solution in $(0, 1)$.

Let $f(x) = x^3 + 3x - 2$

So $f$ is continuous on $[0, 1]$ because $f$ is a polynomial.

$f(0) = -2 < 0$
$f(1) = 2 > 0$

By IVT there is some $x_0$ in $(0, 1)$ such that $f(x_0) = 0$. So $x_0^3 + 3x_0 - 2 = 0$

So $x_0^3 + 3x_0 = 2$
Intermediate Value Theorem: General Case

If $f$ is continuous on $[a, b]$, and $f(a), f(b)$ are on opposite sides of $y = k$, then there is at least one value $x_0$ such that $f(x_0) = k$. 