Tuesday, January 6, 2008

6.2 (cont’d) Some more examples:

(1) \[ \int \sec x (\sec x + \tan x) \, dx \]

\[ = \int (\sec^2 x + \sec x \tan x) \, dx \]

\[ = \int \sec^2 x \, dx + \int \sec x \tan x \, dx \]

\[ = \tan x + \sec x + C \]
(2) \[ \int \tan^2 x \, dx \]

Recall: \[ \tan^2 x + 1 = \sec^2 x \]

\[ = \int (\sec^2 x - 1) \, dx \]
\[ = \int \sec^2 x \, dx - \int 1 \, dx \]
\[ = \tan x - x + C \]
Initial-Value Problems

Recall: \( \int f(x) \, dx = F(x) + C \) represents all of the antiderivatives of \( f(x) \). If we are given more information (an initial value), then we can solve for \( C \) and find a specific antiderivative \( y(x) \) that also satisfies the initial value condition.
Examples: Solve the initial-value problems:

(1) \[ \begin{align*}
\frac{dy}{dx} &= \sqrt[3]{x} - \text{ differential equation} \\
y(1) &= 2
\end{align*} \]

We want a function \( y(x) \) that satisfies both of the above properties.

Given \( \frac{dy}{dx} = \sqrt[3]{x} \) \( \Rightarrow \)

\[ y = \int \sqrt[3]{x} \, dx \]

\[ y = \int x^{\frac{1}{3}} \, dx \]
\[ y = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{3}{4} x^{\frac{4}{3}} + C \]

*Given* \( y(1) = 2 \)

\[ \Rightarrow \quad 2 = \frac{3}{4} (1)^{\frac{4}{3}} + C = \frac{3}{4} + C \]

\[ \Rightarrow \quad C = 2 - \frac{3}{4} = \frac{5}{4} \]

*Solution*: \( y = \frac{3}{4} x^{\frac{4}{3}} + \frac{5}{4} \)
\(2\) \[ \begin{align*} \frac{dy}{dt} &= \frac{3}{\sqrt{1-t^2}} \\ y\left(\frac{\sqrt{3}}{2}\right) &= 0 \end{align*} \]

\[
\frac{dy}{dt} = \frac{3}{\sqrt{1-t^2}} \Rightarrow y = \int \frac{3}{\sqrt{1-t^2}} \, dt
\]

\[
y = 3 \sin^{-1} t + C
\]

\[
y\left(\frac{\sqrt{3}}{2}\right) = 0 \Rightarrow 0 = 3 \sin^{-1} \left(\frac{\sqrt{3}}{2}\right) + C
\]

angle in \([\frac{\pi}{2}, \pi]\) whose sine is \(\frac{\sqrt{3}}{2}\)
\[ 0 = 3 \cdot \frac{\pi}{3} + C = \pi + C \]

\[ C = -\pi \]

Solution: \( y = 3 \sin^{-1} \theta - \pi \)
Chapter 6.3: Integration by Substitution
(or "undoing" the chain rule)

Recall:  $\int u^2 \, du = \frac{1}{3} u^3 + C$

How about $\int \sin^2 x \cos x \, dx$?

Looks complicated, but we can make it as easy as $\int u^2 \, du$ by using the correct substitution.
Let \( u = \sin x \)

\[ u^2 = \sin^2 x \quad \text{and} \quad \frac{du}{dx} = \cos x \Rightarrow du = \cos x \, dx \]

So \( \int \frac{\sin^2 x \cos x \, dx}{u^2} \) becomes \( \int \frac{u^2 \, du}{\frac{1}{3} u^3} = \frac{1}{3} u^3 + C \)
Since the original integral was in terms of $x$, our solution must also be in terms of $x$.

\[ \frac{1}{3} u^3 + C = \frac{1}{3} (\sin x)^3 + C = \frac{1}{3} \sin^3 x + C \]

So \[ \int \sin^2 x \cos x \, dx = \frac{1}{3} \sin^3 x + C \]

Check the answer:

\[ \frac{d}{dx} \left( \frac{1}{3} \sin^3 x + C \right) = \frac{1}{3} \cdot 3 \sin^2 x \cos x = \sin^2 x \cos x \]

by the chain rule
Substitution: The Big Idea

Change the original integral that is in terms of one variable (usually \( x \)) to an equivalent easier integral in terms of a substitution variable (usually \( u \)).

The Big Question: What expression should we choose for \( u \)?

Answer: It takes practice
Recommendation:

Choose an expression that simplifies the integrand and where the derivative \( \frac{du}{dx} \) is also in the integrand.

Some common choices:

- Expressions in \((\_\_\_)'s\)
- Expressions under \(\sqrt{\_\_\_}\)'
- Expressions in denominators
- Expressions in exponents

Caution:

Never let \(u=x\) because you won't change anything!
Examples

(1) \[
\int \frac{1}{x} \sqrt{\ln x} \, dx \quad \text{[} u = \ln x \text{]}
\]

\[
= \int u^{1/2} \, du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3} u^{3/2} + C
\]

\[
= \frac{2}{3} (\ln x)^{3/2} + C
\]
(2) \[ \int (x^4 + 3)^{10} x^3 \, dx \]

**Note:** There is no \( 4x^3 \, dx \) term in the integral, so we multiply and divide by 4

\[ \frac{du}{dx} = 4x^3 \Rightarrow du = 4x^3 \, dx \]

\[ = \frac{1}{4} \int (x^4 + 3)^{10} \cdot 4x^3 \, dx \]

\[ = \frac{1}{4} \int u^{10} \, du \]
\[
= \frac{1}{4} \cdot \frac{1}{11} u'' + C \\
= \frac{1}{44} u'' + C \\
= \frac{1}{44} (x^4 + 3) + C
\]
(3) $\int e^{\tan x} \sec^2 x \, dx$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x \Rightarrow du = \sec^2 x \, dx$

$= \int e^u \, du = e^u + C = e^{\tan x} + C$
(4) \[ \int x^2 \sqrt{x-1} \, dx \]

\[ = \int (u+1)^2 \sqrt{u} \, du \quad \text{[} u = x-1 \text{]} \]

\[ = \int (u^2 + 2u + 1) u^{\frac{1}{2}} \, du \]

\[ = \int (u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}) \, du \]

\[ = \frac{2}{7} u^{\frac{7}{2}} + 2 \cdot \frac{2}{5} u^{\frac{5}{2}} + \frac{2}{3} u^{\frac{3}{2}} + C \]
\[
\begin{align*}
\frac{2}{7} \left( x - 1 \right)^{\frac{7}{2}} + \frac{4}{5} \left( x - 1 \right)^{\frac{5}{2}} + \frac{2}{3} \left( x - 1 \right)^{\frac{3}{2}} + C
\end{align*}
\]