Recall: We would like to find the "area under the curve"
Strategy: Divide the area into $n$ rectangles.

(Overview) Sum up the areas of these rectangles to get an approximation to the area under the curve.
Note as we increase the number of rectangles we get better approximations:

To get the exact area under the curve, we use an infinite number of rectangles, i.e. use $\lim_{n \to \infty}$
Strategy: Divide the interval \([a, b]\) into \(n\) equal subintervals, each with length \(\Delta x = \frac{b-a}{n}\).

These \(n\) subintervals will be the bases of the \(n\) rectangles.
Within the $k^{th}$ subinterval ($k=1,2,...,n$), pick a value $x_k^*$. Let $f(x_k^*)$ be the height of the $k^{th}$ rectangle.

Note: Common choices for $x_k^*$ are the left endpoint, midpoint, or right endpoint of the $k^{th}$ subinterval.
Sum up the areas of the $n$ rectangles:

$$f(x_1^*) \Delta x + f(x_2^*) \Delta x + \ldots + f(x_n^*) \Delta x$$

$$= \sum_{k=1}^{n} f(x_k^*) \Delta x$$

This sum is called a Riemann Sum.

Note: The lower limit of summation in a Riemann Sum will always be 1.

To get the exact area under the curve, we use a limit:

$$\lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*) \Delta x$$
Putting it all together:

Given a function $f(x)$ that is nonnegative and continuous on the interval $[a, b]$ then the area under the curve over $[a, b]$ is

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x$$

where $\Delta x = \frac{b-a}{n}$ is the length of base of each rectangle,

and $f(x_k^*)$ is the value in the $k^{th}$ subinterval used to determine the height of the $k^{th}$ rectangle.
Examples

(1) Compute the area under $f(x) = 6 - x$ over $[1, 5]$.

Use right endpoints.
\[ \Delta x = \frac{5 - 1}{n} = \frac{4}{n} \] length of base of each rectangle

Subintervals: \[ [1, 1 + \frac{4}{n}], [1 + \frac{4}{n}, 1 + \frac{8}{n}], [1 + \frac{8}{n}, 1 + \frac{12}{n}], \ldots \]

Right endpoints: \[ 1 + \frac{4}{n}, 1 + \frac{8}{n}, 1 + \frac{12}{n}, \ldots \]

\[ k = 1 \quad k = 2 \quad k = 3 \quad \ldots \]

The right endpoint of the \( k \)th subinterval is \( 1 + \frac{4k}{n} = x_k^* \)

In general, the right endpoint of the \( k \)th subinterval is \( x_k^* = a + k \Delta x \)
\[ f(x_k^*) = 6 - (1 + \frac{4k}{n}) = \left(5 - \frac{4k}{n}\right) \text{ height of the } k^{th} \text{ rectangle} \]

The sum of the areas of the \( n \) rectangles is

\[ \sum_{k=1}^{n} \left(5 - \frac{4k}{n}\right) \left(\frac{1}{n}\right) = \frac{4}{n} \sum_{k=1}^{n} \left(5 - \frac{4k}{n}\right) = \frac{4}{n} \left[ \sum_{k=1}^{n} 5 - \sum_{k=1}^{n} \frac{4k}{n} \right] \]

\[ = \frac{4}{n} \left[ 5 \sum_{k=1}^{n} 1 - \frac{4}{n} \sum_{k=1}^{n} k \right] = \frac{4}{n} \left[ 5n - \frac{4}{n} \frac{n(n+1)}{2} \right] \]

\[ = \frac{4}{n} \left[ 5n - 2(n+1) \right] = \frac{4}{n} (5n-2n-2) = 4 \left( \frac{3n-2}{n} \right) \text{ approximate area under curve using } n \text{ rect's and right endpoints} \]
The exact area is

\[ A = \lim_{n \to +\infty} 4 \left( \frac{3n-2}{n} \right) \]

\[ = 4 \lim_{n \to +\infty} \frac{3n-2}{n} \]

\[ = 4 \lim_{n \to +\infty} \frac{3n}{n} = 4(3) = 12 \]
(2) Compute the area under \( f(x) = 6 - x \) over \([1, 5]\).

Use left endpoints.

\[
\Delta x = \frac{4}{n}
\]

Subintervals: \([1, 1 + \frac{4}{n}],[1 + \frac{4}{n}, 1 + \frac{8}{n}],[1 + \frac{8}{n}, 1 + \frac{12}{n}],\ldots\)

Left endpoints: \(1, 1 + \frac{4}{n}, 1 + \frac{8}{n}, \ldots\)

\[
k = 1 \quad k = 2 \quad k = 3 \quad \ldots
\]

The left endpoint of the \(k\)th subinterval is \(1 + \frac{4(k-1)}{n}\).

In general, the right endpoint of the \(k\)th subinterval is

\[
X_k^* = a + (k-1)\Delta x
\]
\[ f(x_k^*) = 6 - \left(1 + \frac{4(k-1)}{n}\right) = 5 - \frac{4(k-1)}{n} \]

\[
\frac{1}{n} \sum_{k=1}^{n} f(x_k^*) \Delta x = \frac{1}{n} \sum_{k=1}^{n} \left(5 - \frac{4(k-1)}{n}\right) \left(\frac{y}{n}\right)
\]

\[
= \frac{4}{n} \sum_{k=1}^{n} \left(5 - \frac{4(k-1)}{n}\right) = \frac{4}{n} \left[ \sum_{k=1}^{n} 5 - \sum_{k=1}^{n} \frac{4(k-1)}{n} \right]
\]

\[
= \frac{4}{n} \left[5 \sum_{k=1}^{n} 1 - \frac{4}{n} \sum_{k=1}^{n} (k-1)\right] = \frac{4}{n} \left[5 \sum_{k=1}^{n} 1 - \frac{4}{n} \left(\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1\right)\right]
\]

\[
= \frac{4}{n} \left[5n - \frac{4}{n} \left(\frac{n(n+1)}{2} - n\right)\right] = \frac{4}{n} \left[5n - 2(n+1) + 1\right]
\]
\[
A = \lim_{n \to \infty} \frac{4}{n} \left( \frac{3n+2}{n} \right) = 4 \lim_{n \to \infty} \frac{3n}{n} = 4 \cdot 3 = 12
\]

approximate area under curve using \( n \) rect's and left endpoints.

\[
= \frac{4}{n} \left( 5n - 2n - 2 + 4 \right) = 4 \left( \frac{3n+2}{n} \right)
\]

same exact area as previous problem
3) Compute the area under $f(x) = 6 - x$ over $[1, 5]$ using geometry.

Area of $\Delta = \frac{1}{2}(4)(4) = 8$

Area of $\square = (4)(4) = 16$

Total area $A = 8 + 4 = 12$
Exercises

Compute area under $f(x) = x^2$ over $[0, 1]$.

(4) Use right endpoints
(5) Use left endpoints