Recall: \( f(x), g(x) \) continuous on \([a,b]\)
\( f(x) \geq g(x) \) on \([a,b]\)
The total area between \( f(x) \) and \( g(x) \) on \([a,b]\) is
\[
\int_{a}^{b} (f(x) - g(x)) \, dx
\]
Example: Find the area between $y = \sin x$ and $y = \cos x$ on $[0, 2\pi]$

Intersection:

$\sin x = \cos x$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

Need three integrals

$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x) \, dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x) \, dx + \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x) \, dx = 4\sqrt{2} \quad \text{(Verify)}$$
Chapter 7.2: Volumes

Take a really small vertical rectangle and revolve it about the x-axis.

Case One: Rectangle is touching the x-axis.

Watch "Animate Disk.avi"
The small volume $dV$ of this one arbitrary disk is

$$dV = A(x) \, dx$$

where $A(x)$ is the disk’s cross-sectional area.

Cross-section:

Circle

$$A(x) = \pi (r(x))^2$$

So

$$dV = \pi (r(x))^2 \, dx$$
Case Two: Rectangle is not touching the x-axis

Washer (Annulus)

Watch "AnimateWasher.avi"

Cross-Section:

\[ dV = A(x) \, dx \]

\[ A(x) = \pi (r_2(x))^2 - \pi (r_1(x))^2 = \pi \left[ (r_2(x))^2 - (r_1(x))^2 \right] \]

\[ dV = \pi \left[ (r_2(x))^2 - (r_1(x))^2 \right] \, dx \]
Goal: Find the volume of a solid that results when a region in the $xy$-plane (from $x=a$ to $x=b$) is revolved about the $x$-axis.

Strategy:
- Divide the solid into an infinite number of disks/washers
- Find the small volume $dV$ of one arbitrary disk/washer
- Add up an infinite number of these $dV$'s to get the total volume $V = \int_{a}^{b} dV$
Examples

1) Find the volume of the solid that results when the region under \( f(x) = x^2 \) and over \([0, 1]\) is revolved about the x-axis.

Watch "Animate Volume by Disks.avi"
\[ dV = A(x) \, dx \]

\[ A(x) = \pi \left( r(x) \right)^2 = \pi \left( x^3 \right)^2 = \pi x^4 \]

\[ dV = \pi x^4 \, dx \]

\[ V = \int_0^1 dV = \int_0^1 \pi x^4 \, dx = \frac{\pi}{5} \quad (\text{Verify}) \]

As with total area, total volume must be positive.
(2) Prove that the volume of a sphere with constant radius $R$ is $V = \frac{4}{3}\pi R^3$.

We will revolve the region bounded by the semicircle $y = \sqrt{R^2-x^2}$ and $[-R, R]$ around the $x$-axis.

**Note:** The entire region to be revolved should always be on one side of the axis of revolution, or else we'd get twice the volume.
\[ dV = A(x) \, dx \]

\[ A(x) = \pi \left( \sqrt{R^2 - x^2} \right)^2 = \pi (R^2 - x^2) \]

\[ dV = \pi (R^2 - x^2) \, dx \]

\[ V = \int_{-R}^{R} dV = 2 \int_{0}^{R} dV \quad \text{(by symmetry)} \]

\[ = 2 \pi \int_{0}^{R} (R^2 - x^2) \, dx = 2 \pi \left( R^2 x \bigg|_{0}^{R} - \frac{1}{3} x^3 \bigg|_{0}^{R} \right) \]

\[ = 2 \pi \left( R^3 - \frac{1}{3} R^3 \right) = 2 \pi \left( \frac{2}{3} R^3 \right) = \frac{4}{3} \pi R^3 \]
(3) Region: Bounded by \( y = x^2 \) and \( y = \sqrt{x} \)
Revolve about the x-axis

\[
\begin{align*}
\text{made up of an infinite number of really small washers}\\
\text{One arbitrary washer:} \\
\text{Watch} "\text{Animate Volume by Washers.avi}" \\
\end{align*}
\]
\[ dV = A_l(x) \, dx \]

\[ A_l(x) = \pi \left[ (r_l(x))^2 - (r_r(x))^2 \right] \]

\[ A_l(x) = \pi \left[ (\sqrt{x})^2 - (x^2)^2 \right] = \pi \left[ x - x^4 \right] \]

\[ dV = \pi (x-x^4) \, dx \]

\[ V = \int_0^1 dV = \int_0^1 \pi (x-x^4) \, dx = \frac{3\pi}{10} \]
To revolve about the y-axis we need horizontal rectangles dy.

Region: Bounded by $y = x^2$, $y = 2x$

Revolve about y-axis.

Washer
\[
\begin{align*}
dV &= A(y) \, dy \\
A(y) &= \pi \left[ (\sqrt{y})^2 - \left( \frac{1}{2} y \right)^2 \right] \\
dV &= \pi \left( y - \frac{1}{4} y^2 \right) \, dy \\
V &= \int_0^y dV = \int_0^y \pi \left( y - \frac{1}{4} y^2 \right) dy = \frac{8\pi}{3} \\
\text{Intersection} \\
\sqrt{y} &= \frac{1}{2} y \\
y &= \frac{1}{4} y^2 \\
y_1 y &= y^2 \\
0 &= y^2 - y_1 y = y(y - y) \\
y &= 0, y = 4
\end{align*}
\]
Revolving about other horizontal/vertical lines

(5) Region: \( y = x^2, \ x = 0, \ y = 16 \)

Revolve about \( y = 16 \)

\[
\begin{align*}
&\text{Disk} \\
&r(x) = 16 - x^2
\end{align*}
\]
\[ dV = A(x) \, dx \]

\[ A(x) = \pi (16-x^2)^2 \]

\[ dV = \pi (16-x^2)^2 \, dx \]

\[ V = \int_0^4 \pi (16-x^2)^2 \, dx \]

\[ V = \int_0^4 \pi (256 - 32x^2 + x^4) \, dx = \frac{8192\pi}{15} \quad (\text{Verify}) \]
(b) Region: \( y = x^2, \ x = 0, \ y = 16 \)

Revolve about \( x = 4 \)

\[ r_1(y) = 4 - \sqrt{y} \]

\[ r_2(y) = 4 \]

\[ dy \]

Washer
\[ dV = A(y) \, dy \]
\[ A(y) = \pi \left[ y^2 - (4 - \sqrt{y})^2 \right] \, dy \]
\[ dV = \pi \left[ 16 - (4 - \sqrt{y})^2 \right] \, dy \]
\[ V = \int_0^{16} dV = \int_0^{16} \pi \left[ 16 - (16 - 8\sqrt{y} + y) \right] \, dy \]
\[ V = \int_0^{16} \pi \left( 8\sqrt{y} - y \right) \, dy = \frac{640\pi}{3} \quad \text{(Verify)} \]
Not all solids are solids of revolution, i.e. have cross-sections made up of circles.

E.g.

- horizontal cross-section is a square
- vertical cross-section is a triangle

To find the volumes of these solids, we still use the same strategy.
(7) Find the volume of the solid whose base is the region between \( y = x^3 \) and the \( x \)-axis from \( x = 0 \) to \( x = 2 \) and whose cross sections perpendicular to the \( x \)-axis are squares.

Full solid is made up of an infinite number of these small square wedges.
The small volume of an arbitrary wedge is

\[ dV = A(x) \, dx \]

\[ A(x) = (x^3)^2 \]

\[ dV = x^6 \, dx \]

\[ V = \int_{0}^{2} dV = \int_{0}^{2} x^6 \, dx = \frac{128}{7} \quad \text{(Verify)} \]
(8) Base: Region enclosed by \( y = \sqrt{x}, y = 0, x = 4 \)
Cross sections \( \perp \) x-axis are semicircles with diameter across the base.

\[
dV = A(x) \, dx \\
A(x) = \frac{1}{2} \pi \left( \frac{1}{2} \sqrt{x} \right)^2 \\
dV = \frac{1}{8} \pi x \, dx \\
V = \int_0^4 \frac{1}{8} \pi x \, dx = 11 \\
(Verify)
\]