Chapter 9.1: Intro to Differential Equations

Definition: An ordinary differential equation (ODE) is an equation that includes the derivative of some unknown function. The order of an ODE is the order of the highest order derivative in the equation, e.g.

\[ \frac{dy}{dx} = x \sqrt{x^2 + y} \quad \text{1st order} \]

\[ xy' = (1-x)y \quad \text{1st order} \]

\[ \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 8y = 0 \quad \text{2nd order} \]

\[ m \frac{d^2y}{dt^2} = -mg \quad \text{2nd order} \, \text{free fall model} \]

\[ \frac{d^2y}{dt^2} + \frac{c}{m} \frac{dy}{dt} = -g \quad \text{2nd order} \, \text{free fall subject to air resistance} \]
The solution to an ODE is a function that satisfies the equation.

Example: Verify that $y = Ae^{2x} + Be^{-4x}$ ($A, B$ constants) is a solution to the ODE $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0$.

If $y = Ae^{2x} + Be^{-4x}$

$\Rightarrow \frac{dy}{dx} = 2Ae^{2x} - 4Be^{-4x}$ and $\frac{d^2y}{dx^2} = 4Ae^{2x} + 16Be^{-4x}$

So $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y$

$= 4Ae^{2x} + 16Be^{-4x} + 2(2Ae^{2x} - 4Be^{-4x}) - 8(Ae^{2x} + Be^{-4x})$
$= 4Ae^{2x} + 16Be^{-4x} + 4Ae^{2x} - 8Be^{-4x} - 8Ae^{2x} - 8Be^{-4x}$
$= 8Ae^{2x} - 8Ae^{2x} + 16Be^{-4x} - 16Be^{-4x}$

$= 0 \checkmark$
Practice: Verify that \( y = xe^{-x} \) is a solution to \( xy' = (1-x)y \)

Solving ODE's

(1) First-Order Linear Equations

Form: \( \frac{dy}{dx} + p(x)y = q(x) \)

(a) Special Case: \( p(x) = 0 \), i.e. \( \frac{dy}{dx} = q(x) \)

E.g. to solve \( \frac{dy}{dx} = x\sqrt{x^2+y} \)

we just integrate both sides with respect to (wrt) \( x \)

\[
y = \int x\sqrt{x^2+y} \, dx \quad \text{u-sub: } u = x^2 + y
\]

\[
y = \frac{1}{3} (x^2 + y)^{\frac{3}{2}} + C \quad \text{(Show details)}
\]
Recall how to solve initial-value problems:

\[ \begin{cases} \frac{dy}{dx} = x \sqrt{x^2+y} \\ y(-4) = 0 \end{cases} \]

\[ y = \frac{1}{8} (x^2+y)^{\frac{3}{2}} + C \]

\[ 0 = \frac{1}{8} (-4^2+4)^{\frac{3}{2}} + C = \frac{1}{8} (20)^{\frac{3}{2}} + C \]

\[ C = \frac{-40\sqrt{5}}{3} \]

Solution: \[ y = \frac{1}{3} (x^2+y)^{\frac{3}{2}} - \frac{40\sqrt{5}}{3} \]
(b) \( p(x) \neq 0 \)

e.g. \[ \frac{dy}{dx} + 3y = e^{-2x} \], i.e. \( p(x) = 3, q(x) = e^{-2x} \)

Multiply both sides by

\[ \mu(x) = e^{3x} \] (Later we'll see where this comes from)

\[ e^{3x} \frac{dy}{dx} + y(3e^{3x}) = e^{-2x} e^{3x} = e^x \]

Result of a product rule

\[ \frac{d}{dx} (e^{3x} y) = e^x \]

Now integrate both sides wrt \( x \)

\[ \int \frac{d}{dx} (e^{3x} y) \, dx = \int e^x \, dx \]
\[ e^{3x}y = e^x + C \quad \text{Note: We only need one } + C, \text{ usually on RHS} \]

\[ \frac{e^x}{e^{3x}} + \frac{C}{e^{3x}} = e^{-2x} + Ce^{-3x} \]

Check solution: \( \frac{dy}{dx} + 3y = e^{-2x} \)

\[ y = e^{-2x} + Ce^{-3x} \Rightarrow \frac{dy}{dx} = -2e^{-2x} - 3Ce^{-3x} \]

\[ \frac{dy}{dx} + 3y = -2e^{-2x} - 3Ce^{-3x} + 3(e^{-2x} + Ce^{-3x}) \]

\[ = -2Ce^{-2x} - 3Ce^{-3x} + 3e^{-2x} + 3Ce^{-3x} \]

\[ = e^{-2x} \quad \checkmark \]
Now where did \( \mu(x) = e^{3x} \) come from?  \( \mu(x) \) is called the integrating factor of the ODE.

We always let \( \mu(x) = e^{\int p(x) \, dx} \), so

if \( p(x) = 3 \) then \( \mu(x) = e^{\int 3 \, dx} = e^{3x} \)

Note: Don't need +C here since we can choose any antiderivative of \( p(x) \).

Claim: If we multiply both sides of \( \frac{dy}{dx} + p(x)y = q(x) \) by \( \mu(x) = e^{\int p(x) \, dx} \), the LHS will be the result of a product rule, namely \( \frac{d}{dx}(\mu y) \).

Here's why: Let \( \mu(x) = e^{\int p(x) \, dx} \)

Note that \( \frac{d}{dx} = e^{\int p(x) \, dx} \cdot \frac{d}{dx}(\int p(x) \, dx) = e^{\int p(x) \, dx} \cdot p(x) \)

\( \frac{d}{dx} = \mu p(x) \)

Now multiply \( \frac{dy}{dx} + p(x)y = q(x) \) by \( \mu(x) \) to get
\[
\begin{align*}
\mu \frac{dy}{dx} + \mu p(x)y &= \mu q(x) \\
\mu \frac{dy}{dx} + \frac{d\mu}{dx} y &= \mu q(x) \\
\text{Result of product rule} \\
\frac{d}{dx} (\mu y) &= \mu q(x)
\end{align*}
\]

Now we would integrate both sides w.r.t. \(x\) and solve for \(y\).

**Example:** Solve the initial-value problem

\[
\begin{cases}
\quad x^2 \frac{dy}{dx} + xy = x \sin x , \quad x > 0 \\
\quad y(\pi) = 2
\end{cases}
\]
\[ x^2 \frac{dy}{dx} + xy = x \sin x \]

Divide by \( x^2 \) (\( \neq 0 \) because \( x > 0 \))

\[ \frac{dy}{dx} + \frac{1}{x} y = \frac{\sin x}{x} \quad p(x) = \frac{1}{x}, \quad q(x) = \frac{\sin x}{x} \]

\[ \mu(x) = e^{\int \frac{1}{x} \, dx} = e^{\ln|x|} = e^{\ln x} = x \]

So

\[ x \frac{dy}{dx} + y = \sin x \]

\[ \frac{d}{dx} (xy) = \sin x \]

\[ \int \frac{d}{dx} (xy) \, dx = \int \sin x \, dx \]

\[ xy = -\cos x + C \]
\[ y = \frac{-\cos x}{x} + \frac{C}{x} \]

Now \[ y(\pi) = 2 \]

\[ 2 = -\frac{\cos \pi}{\pi} + \frac{C}{\pi} = -\frac{1}{\pi} + \frac{C}{\pi} = \frac{1 + C}{\pi} \]

\[ 1 + C = 2\pi \implies C = 2\pi - 1 \]

Solution: \[ y = \frac{-\cos x}{x} + \frac{2\pi - 1}{x} \]
(2) First-Order Separable Equations

Form: \( h(y) \frac{dy}{dx} = g(x) \)

We can "separate" the variables to either side

\( h(y) \, dy = g(x) \, dx \) \hspace{1cm} \text{Differential form}

Now integrate both sides (LHS wrt y, RHS wrt x)

\[ \int h(y) \, dy = \int g(x) \, dx \]

\( H(y) = G(x) + C \) \hspace{1cm} \text{where} \quad H'(y) = h(y) \quad \text{and} \quad G'(x) = g(x) \)

This is an implicit solution of the ODE. If we can, we should solve for \( y \) explicitly.
Example

Solve: \[ \frac{dy}{dx} = 2(1+y^2) \times \]
\[ \frac{dy}{1+y^2} = 2x \, dx \quad \text{Note: } 1+y^2 \neq 0 \]
\[ \int \frac{dy}{1+y^2} = \int 2x \, dx \]
\[ \tan^{-1} y = x^2 + C \]
\[ y = \tan(x^2 + C) \]

Check: \( y = \tan(x^2 + C) \Rightarrow \)
\[ \frac{dy}{dx} = \sec^2 (x^2 + C) \cdot \frac{d}{dx} (x^2 + C) \Rightarrow \frac{dy}{dx} = 2x \sec^2 (x^2 + C) \]

RHS of ODE: \[ 2x (1+y^2) = 2x (1+\tan^2 (x^2+C)) \]
\[ = 2x \sec^2 (x^2 + C) \]
Example:

Solve: \( \frac{dy}{dx} = -xy \)

If \( y \neq 0 \): \( \frac{dy}{y} = -x \, dx \)

\( \int \frac{dy}{y} = \int -x \, dx \)

\( \ln |y| = -\frac{1}{2} x^2 + C \)

\( |y| = e^{-\frac{1}{2} x^2 + C} = e^{-\frac{1}{2} x^2} e^C \)

\( |y| = C e^{-\frac{1}{2} x^2} \)

\( y = \pm Ce^{-\frac{1}{2} x^2} \)

Note: If \( C = 0 \) then \( y = 0 \), so it is not necessary to write "and also \( y = 0 \)" as part of our answer.

Note: \( y = 0 \) is a solution to the ODE.
Example: Find a curve in the $xy$-plane that passes through $(0, 3)$ and whose tangent line at any point $(x, y)$ has slope $\frac{2x}{y^2}$. So we need to solve the initial-value problem

\[
\left\{
\begin{array}{l}
\frac{dy}{dx} = \frac{2x}{y^2} \\
y(0) = 3
\end{array}
\right.
\]

\[
\frac{dy}{dx} = \frac{2x}{y^2}
\]

\[
y^2 \, dy = 2x \, dx
\]

\[
\int y^2 \, dy = \int 2x \, dx
\]

\[
\frac{1}{3} y^3 = x^2 + C
\]
Now $y(0) = 3$

so $\frac{1}{3} (3)^3 = 0^2 + C \Rightarrow C = 9$

$\frac{1}{3} y^3 = x^2 + 9$

$y^3 = 3x^2 + 27$

$y = \sqrt[3]{3x^2 + 27}$