Chapter 8.8: Improper Integrals

Motivation: What is wrong with the following "solution"?

\[
\int_0^2 \frac{dx}{(x-1)^2} = \int_0^2 (x-1)^{-2} \, dx = \left. \frac{(x-1)^{-1}}{-1} \right|_0^2 = -\frac{1}{x-1} \bigg|_0^2
\]

\[
= -\left( \frac{1}{2-1} - \frac{1}{0-1} \right) = -(1-(-1)) = -2
\]

But the graph of \( f(x) = \frac{1}{(x-1)^2} \) is above the x-axis, so the net-signed area can't be negative.

The problem is that \( f(x) = \frac{1}{(x-1)^2} \) is not continuous on \([0, 2]\), so we can't use the Fundamental Theorem of Calculus.

\( \int_0^2 \frac{dx}{(x-1)^2} \) is called an improper integral.
Another type of improper integral is easier to recognize because at least one of the limits of integration is infinite.

Examples

(1) \[ \int_{1}^{\infty} \frac{1}{x^2} \, dx \]  
   To "fix" this improper integral, we introduce a limit.

\[ \lim_{t \to \infty} \int_{1}^{t} \frac{1}{x^2} \, dx \]

Caution: Only the limit of integration changes. The integrand and differential remain the same.

Now \[ \int_{1}^{t} \frac{1}{x^2} \, dx = -\frac{1}{x} \bigg|_{1}^{t} = -\left( \frac{1}{t} - \frac{1}{1} \right) = -\frac{1}{t} + 1 \]

and \[ \lim_{t \to \infty} \left(-\frac{1}{t} + 1 \right) = 1. \]

We say \( \int_{1}^{\infty} \frac{1}{x^2} \, dx \) converges to 1.
(2) \[ \int_{1}^{+\infty} \frac{1}{x} \, dx = \lim_{t \to +\infty} \int_{1}^{t} \frac{1}{x} \, dx \]

Now \[ \int_{1}^{t} \frac{1}{x} \, dx = \ln|x| \bigg|_{1}^{t} = \ln|t| - \ln|1| = \ln|t| \]

and \[ \lim_{t \to +\infty} \ln|t| = +\infty \]

Since limit does not exist, we say \[ \int_{1}^{+\infty} \frac{1}{x} \, dx \text{ diverges} \]
(3) \[ \int_{-\infty}^{0} \frac{1}{1+x^2} \, dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1+x^2} \, dx \]

Now \[ \int_{t}^{0} \frac{1}{1+x^2} \, dx = \tan^{-1} x \bigg|_{t}^{0} = \tan^{-1} 0 - \tan^{-1} t = -\tan^{-1} t \]

and \[ \lim_{t \to -\infty} -\tan^{-1} t = -(-\frac{\pi}{2}) = \frac{\pi}{2} \]

Recall.
\[
\int_{-\infty}^{+\infty} \frac{1}{1+x^2} \, dx
\]

We must split this into two integrals and solve each separately:

\[
= \int_{-\infty}^{0} \frac{1}{1+x^2} \, dx + \int_{0}^{+\infty} \frac{1}{1+x^2} \, dx
\]

\[
\frac{\pi}{2} \quad (\text{previous example}) \quad \frac{\pi}{2} \quad (\text{Practice: Show work})
\]

Since both integrals converged,

\[
\int_{-\infty}^{+\infty} \frac{1}{1+x^2} \, dx \text{ converges to } \frac{\pi}{2} + \frac{\pi}{2} = \pi
\]
Note: If at least one of the integrals diverges, then the original integral diverges, e.g.
\[ \int_{-\infty}^{\infty} x \, dx \text{ diverges because } \int_{-\infty}^{0} x \, dx \text{ diverges} \]

Caution: When both limits of integration are infinite, you must split up the integral.
\[ \int_{-\infty}^{\infty} x \, dx \neq \lim_{t \to \infty} \int_{-t}^{t} x \, dx \]
This will give you the wrong answer.
Now we return to integrals $\int_a^b f(x) \, dx$ where $f(x)$ has a discontinuity on $[a, b]$.

(5) \[ \int_0^1 \frac{dx}{\sqrt{1-x}} \] Note: $f(x) = \frac{1}{\sqrt{1-x}}$ is not continuous at $x=1$.

= \lim_{t \to 1^-} \int_0^t \frac{dx}{\sqrt{1-x}} \quad u = 1-x \quad x = t \Rightarrow u = 1-t
\quad du = -dx \quad x = 0 \Rightarrow u = 1

(Since $1-x$ is linear, you could also do a 'quick' substitution)

So \[ -\int_0^t \frac{du}{\sqrt{u}} = -2u^{\frac{1}{2}} \bigg|_0^{1-t} = -2 \left( \sqrt{1-t} - 1 \right) \]

and \[ \lim_{t \to 1^-} -2 \left( \sqrt{1-t} - 1 \right) = 2 \] so \[ \int_0^1 \frac{dx}{\sqrt{1-x}} \] converges to 2
(b) \[ \int_{0}^{1} \frac{dx}{1-x} = \lim_{t \to 1^-} \int_{0}^{t} \frac{dx}{1-x} \]

Now \[ \int_{0}^{t} \frac{dx}{1-x} = -\ln |1-x| \bigg|_{0}^{t} = -\left( \ln |1-t| - \ln 1 \right) \]

\[ = -\ln |1-t| \]

Note as \( t \to 1^- \), \( |1-t| \to 0^+ \) so \( \ln |1-t| \to -\infty \)

So \( \lim_{t \to 1^-} -\ln |1-t| = -(-\infty) = +\infty \)

So \( \int_{0}^{1} \frac{dx}{1-x} \) diverges.
\[
\int_{1}^{4} \frac{dx}{(x-2)^{2/3}} = \int_{1}^{2} \frac{dx}{(x-2)^{2/3}} + \int_{2}^{4} \frac{dx}{(x-2)^{2/3}}
\]

Since the discontinuity is between \( a=1 \) and \( b=4 \), we must split this up at the point of discontinuity.

\( \int_{1}^{2} \frac{dx}{(x-2)^{2/3}} \) converges to 3 (practice: verify)

we'll do this one now

\[\int_{2}^{4} \frac{dx}{(x-2)^{2/3}}\]

Solve each separately

\[\int_{2}^{4} \frac{dx}{(x-2)^{2/3}} = \lim_{t \to 2^+} \int_{t}^{4} \frac{dx}{(x-2)^{2/3}} \quad \text{(I'll do a 'quick' substitution)}\]

Now \( \int_{t}^{4} (x-2)^{-2/3} \, dx = 3 \left( x-2 \right)^{1/2} \bigg|_{t}^{4} \)
\[ = 3 \left( 3\sqrt{2} - 3\sqrt{6-2} \right) \]

and \( \lim_{b \to 2^+} 3 \left( 3\sqrt{2} - 3\sqrt{b-2} \right) = 3\sqrt{2} \)

So \( \int_0^4 \frac{dx}{(x-2)^{2/3}} \) converges to \( 3 + 3\sqrt{2} \)

Exercise: Compute \( \int_0^2 \frac{dx}{(x-1)^2} \)

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(8) Find the area between the x-axis and

\[ y = \frac{8}{x^2 - y} \quad \text{for} \quad x \geq y \]

\[ \int_{y}^{\infty} \frac{8}{x^2 - y} \, dx = \lim_{t \to +\infty} \int_{y}^{t} \frac{8}{x^2 - y} \, dx \quad \text{Use partial fractions} \]

\[ \frac{8}{x^2 - y} = \frac{8}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \]

\[ 8 = A(x+2) + B(x-2) \]

\[ x = -2: \quad 8 = 0 + B(-4) \quad \Rightarrow \quad B = -2 \]

\[ x = 2: \quad 8 = A(4) + 0 \quad \Rightarrow \quad A = 2 \]

\[ \int_{y}^{t} \frac{8}{x^2 - y} \, dx = \int_{y}^{t} \frac{2}{x-2} \, dx + \int_{y}^{t} \frac{-2}{x+2} \, dx \]
\[
\begin{align*}
&= 2 \ln |x-2| \bigg|_y^t - 2 \ln |x+2| \bigg|_y^t \\
&= 2 \left( \ln |t-2| - \ln 2 \right) - 2 \left( \ln |t+2| - \ln 6 \right) \\
&= 2 \ln |t-2| - 2 \ln |t+2| - 2 \ln 2 + 2 \ln 6 \\
&= 2 \ln \left| \frac{t-2}{t+2} \right| - 2 \ln 2 + 2 \ln 6
\end{align*}
\]

So

\[
\lim_{t \to +\infty} \left( 2 \ln \left| \frac{t-2}{t+2} \right| - 2 \ln 2 + 2 \ln 6 \right)
\]

\[
= 2 \ln 1 - 2 \ln 2 + 2 \ln 6
\]

\[
= 2 \ln 6 - 2 \ln 2 = \ln 36 - \ln 4 = \ln 9
\]