Recall: To find the slope of the tangent line to \( y = f(x) \) at \( x = x_0 \), we compute \( \frac{dy}{dx} \bigg|_{x=x_0} \).

E.g., the slope of the tangent line to \( y = x^2 + x \) at \( x = 2 \):

\[
\frac{dy}{dx} = 2x + 1 \quad \Rightarrow \quad \frac{dy}{dx} \bigg|_{x=2} = 2(2) + 1 = 5
\]
Curves in the xy-plane can be described in parametric form by giving both $x$ and $y$ as functions of some parameter $t$.

*E.g.* $x = 2 \sin t, \quad y = 4 \cos t \quad 0 \leq t \leq 2\pi$

Think of $t$ as time and the curve as the path of an object in the xy-plane.

So $t=0$: $x = 2 \sin 0 = 0$, $y = 4 \cos 0 = 4$

At $t=0$, the object is at $(0, 4)$

To find the slope of the tangent line to a parametric curve $x = f(t), \quad y = g(t) \quad a \leq t \leq b$

At $t = t_0$, we compute

$$\left. \frac{dy}{dx} \right|_{t = t_0} \quad \text{where} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$
Example: Find the slope of the tangent line to
\[ x = 2 \sin t, \quad y = 4 \cos t \quad 0 \leq t \leq 2\pi \]

at \( t = \frac{\pi}{4} \).

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-4 \sin t}{2 \cos t} = -2 \tan t
\]

\[
\left. \frac{dy}{dx} \right|_{t=\frac{\pi}{4}} = -2 \tan \frac{\pi}{4} = -2
\]

To graph \( x = 2 \sin t, \quad y = 4 \cos t \quad 0 \leq t \leq 2\pi \)
we can eliminate the parameter by using the identity
\[ \sin^2 t + \cos^2 t = 1 \]
Now \( \sin t = \frac{x}{2} \) and \( \cos t = \frac{y}{4} \)
So \( \left( \frac{x}{2} \right)^2 + \left( \frac{y}{4} \right)^2 = 1 \)

\[
\frac{x^2}{4} + \frac{y^2}{16} = 1
\]

which is the ellipse:

\[
x = 2 \sin \frac{\pi}{4} = \sqrt{2} \\
y = 4 \cos \frac{\pi}{4} = 2 \sqrt{2}
\]

Slope of tangent line is \(-2\)

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**Horizontal and Vertical Lines**

Since \( \frac{dy}{dx} = \frac{dy}{dt} \) we have

**Horizontal Tangent Line** \(\iff\) \( \frac{dy}{dx} = 0 \) \(\iff\) \( \frac{dy}{dt} = 0 \)

**Vertical Tangent Line** \(\iff\) \( \frac{dy}{dx} \) undefined \(\iff\) \( \frac{dx}{dt} = 0 \)

**Note:** If \( \frac{dy}{dt} = \frac{dx}{dt} = 0 \) at \( t = b \), the point is a **singular point**.

No conclusion can be made. It's best to look at the graph in these cases.
Tangent Lines for Polar Curves

The slope of the tangent line to a polar curve \( r = f(\theta) \) at \( \theta = \theta_0 \) is

\[
\frac{dy}{dx} \bigg|_{\theta = \theta_0} = \text{where } \frac{dy}{d\theta} = \frac{\frac{dy}{dx}}{\frac{dx}{d\theta}}
\]

and we use \( y = r \sin \theta \), \( x = r \cos \theta \) to compute \( \frac{dy}{d\theta}, \frac{dx}{d\theta} \)

\[
y = r \sin \theta \Rightarrow \frac{dy}{d\theta} = \]

\[
x = r \cos \theta \Rightarrow \frac{dx}{d\theta} = \]

So \( \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{r(-\sin \theta) + \cos \theta \frac{dr}{d\theta}} \)
Example: Find the slope of the tangent line to \( r = 1 + \cos \theta \) at \( \theta = \frac{\pi}{2} \).

\[
\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{r \cos \theta - \sin \theta \frac{dr}{d\theta}}
\]

\( r = 1 + \cos \theta \Rightarrow \frac{dr}{d\theta} = -\sin \theta \)

So \( \frac{dy}{dx} = \frac{(1 + \cos \theta)(\cos \theta) + (\sin \theta)(-\sin \theta)}{(1 + \cos \theta)(-\sin \theta) + (\cos \theta)(-\sin \theta)} \)

Recall \( \sin \frac{\pi}{2} = 1, \cos \frac{\pi}{2} = 0 \)

so \( \frac{dy}{dx} \bigg|_{\theta = \frac{\pi}{2}} = \frac{(1 + 0)(1) + (1)(-1)}{(1 + 0)(0) + (0)(1)} = \frac{-1}{1} = -1 \)
Graph: \( r = 1 + \cos \theta \) is a cardioid.

(1, \( \frac{\pi}{2} \))

- Slope of tangent line is 1.