Chapter 10.2: Polar Coordinates

Rectangular Coordinates

![Rectangular Coordinate System]

Polar Coordinates:

![Polar Coordinate System]
With polar coordinates a point $P$ does not have a unique $(r, \theta)$-pair.

E.g. $(1, \frac{\pi}{4})$

Can also be described as

$(1, \frac{\pi}{4} + 2\pi)$

Or as $(1, \frac{\pi}{4} - 2\pi)$ (so $\theta$ can be negative)

Or $(1, \frac{\pi}{4} + 2\pi k)$ for any integer $k$
Also, $r$ can be negative,

e.g. $(-1, \frac{5\pi}{4})$

When $r < 0$, we go $1r$ units in the "opposite" direction of $\theta$, i.e. $\theta \pm \pi$

Practice: Describe $P$ with two other $(r, \theta)$ pairs

Note: this is same point $P$ from earlier.
Transformation Equations

Polar $\Rightarrow$ Rectangular

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

Rectangular $\Rightarrow$ Polar

\[ r^2 = x^2 + y^2 \]
\[ \tan \theta = \frac{y}{x} \]

Example: Convert \( (7, \frac{2\pi}{3}) \) to rectangular coordinates

\[ x = 7 \cos \frac{2\pi}{3} = 7 \left(-\cos \frac{\pi}{2}\right) = 7 \left(-\frac{1}{2}\right) = -\frac{7}{2} \]
\[ y = 7 \sin \frac{2\pi}{3} = 7 \left(\sin \frac{\pi}{3}\right) = 7 \left(\frac{\sqrt{3}}{2}\right) = \frac{7\sqrt{3}}{2} \]

So \( (7, \frac{2\pi}{3}) = \left(-\frac{7}{2}, \frac{7\sqrt{3}}{2}\right) \)
Example: Convert \((-8, -8)\) to polar coordinates

Assume \(r \geq 0\) and give two solutions:

1. \(0 \leq \theta < 2\pi\)
2. \(-2\pi \leq \theta < 0\)

\[r = \sqrt{(-8)^2 + (-8)^2} = \sqrt{64+64} = 8\sqrt{2}\]

\[\tan \theta = \frac{-8}{-8} = 1\]

Note: \(\tan \frac{\pi}{4} = 1\) but since \(r \geq 0\) that would give a point in first quadrant but \((-8, -8)\) is in the third quadrant.

So \(\theta = \frac{5\pi}{4}\) or \(\theta = -\frac{3\pi}{4}\)

Solutions:

1. \((8\sqrt{2}, \frac{5\pi}{4})\)
2. \((8\sqrt{2}, -\frac{3\pi}{4})\)
Polar Curves (usually of form \( r = f(\theta) \))

Note: See pre-printed notes for more details

**Examples**

(1) \( r = 5 \)  
Circle
  
  Center: \((0,0)\)  
  Radius: 5

Convert to rectangular coordinates

\[ r = 5 \implies r^2 = 5^2 \implies x^2 + y^2 = 5^2 \]

In general, \( r = a \) is a circle with
  
  Center \((0,0)\) and a radius of \(a\)

One revolution: \( 0 \leq \theta \leq 2\pi \)

(or any interval of length \(2\pi\))
(2) \( \theta = \frac{\pi}{6} \) Line

In general, \( \theta = \theta_0 \) is a line

\[
\begin{align*}
\theta &= \frac{\pi}{6} \\
\theta &= \theta_0
\end{align*}
\]

(3) \( r = 3 \sin \theta \) (see pre-printed notes)

In general, \( r = \pm a \sin \theta \), \( r = \pm a \cos \theta \) \((a > 0)\) are circles but only need \( 0 \leq \theta \leq \pi \) (or any interval of length \( \pi \)) for one revolution.

\[
\begin{align*}
\text{a} & \quad \text{a} & \quad \text{a} & \quad \text{a} \\
\text{r} = a \sin \theta & \quad \text{r} = -a \sin \theta & \quad \text{r} = a \cos \theta & \quad \text{r} = -a \cos \theta
\end{align*}
\]
(4) $r = 2 - 2\cos \theta = 2(1 - \cos \theta)$

(see pre-printed notes)

In general, $r = a(1 \pm \cos \theta)$, $r = a(1 \pm \sin \theta)$ are **cardioids**. One revolution: $0 \leq \theta \leq 2\pi$

\[ r = a(1 + \cos \theta) \quad r = a(1 - \cos \theta) \]

\[ r = a(1 + \sin \theta) \quad r = a(1 - \sin \theta) \]
Cardioids are a special case of
\[ r = a \pm b \cos \theta, \quad r = a \pm b \sin \theta \quad (a, b > 0) \]
where \( a = b \). These curves are called \textit{limacons}.

\[ r = 1 + 2 \cos \theta \]

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( \pi )</td>
<td>-1</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>1</td>
</tr>
</tbody>
</table>

\( r = 1 + 2 \cos \theta \)
(5) \( r = 4 \cos 2\theta \) (see pre-printed notes)

In general, \( r = a \cos n\theta \), \( r = a \sin n\theta \) \((a > 0, \ n \geq 2)\) are roses.

\( n \) odd \( \Rightarrow \) \( n \) petals
\( 0 \leq \theta \leq \pi \) for one revolution

\( n \) even \( \Rightarrow \) \( 2n \) petals
\( 0 \leq \theta \leq 2\pi \) for one revolution
Additional Example:

Sketch \( r = 2\sin 3\theta \)

Rose with 3 petals, requires \( 0 \leq \theta \leq \pi \) for one full trace

Find "tips" of petals:

\[
\sin 3\theta = 1, \quad \sin 3\theta = -1
\]

\[
3\theta = \frac{\pi}{2}, \frac{5\pi}{2} \quad 3\theta = \frac{3\pi}{2}
\]

\[
\theta = \frac{\pi}{6}, \frac{5\pi}{6} \quad \theta = \frac{3\pi}{6} = \frac{\pi}{2}
\]