Chapter 5.4: The Definition of Area as a Limit

Goal: Given a f(x) that is nonnegative and continuous on \([a,b]\), find the "area under the curve."

\[ \sum_{k=1}^{10} k^2 \]

Preliminaries: Sigma Notation
- A concise way of denoting sums

\( \sum_{k=1}^{10} k^2 \) \( \quad \sum \) - upper limit of summation
\( k \) \( \quad \sum \) - a function of the index variable
\( k = 1 \) \( \quad \sum \) - lower limit of summation
\( \Rightarrow \) \( \quad \sum \) - index variable

"Sum as k goes from 1 to 10 of \( k^2 \)"
Examples

(1) \[ \sum_{k=1}^{50} (2k-1) = (2(1)-1) + (2(2)-1) + (2(3)-1) + \ldots + (2(50)-1) = 1 + 3 + 5 + \ldots + 99 \]

(2) \[ \sum_{k=-2}^{2} \sin(k \pi) = \sin(-2 \pi) + \sin(-\pi) + \sin(0) + \sin(\pi) + \sin(2 \pi) = 0 \]

(3) Express \( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{20} \) in sigma notation
\[ \sum_{k=1}^{20} \frac{1}{k} \]

(4) Express \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{19} - \frac{1}{20} \) in sigma notation
\[ \sum_{k=1}^{20} \left( -1 \right)^{k+1} \frac{1}{k} \]
Note: The name of the index variable is unimportant.

\[
e_{q} \sum_{k=1}^{20} \frac{(-1)^{k+1}}{k} = \sum_{i=1}^{20} \frac{(-1)^{i+1}}{i}
\]

Some important summation formulas:
(The lower index of summation must be 1)

1) \[1 + 2 + 3 + \ldots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}
\]

- open form
- closed form

e.g. \[1 + 2 + 3 + \ldots + 100 = \frac{100(101)}{2} = 5050\]
\[ (2) \quad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \]

\[ (3) \quad 1^3 + 2^3 + 3^3 + \cdots + n^3 = \sum_{k=1}^{n} k^3 = \left[ \frac{n(n+1)}{2} \right]^2 \]

\[ (4) \quad \sum_{k=1}^{n} 1 = n \]

Properties of summations (lower index need not be 1)

1. \[ \sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k \quad \text{for any constant } c \text{ that does not depend on } k \]

2. \[ \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \]
Example: Evaluate \( \sum_{k=1}^{50} (2k-1) \)

\[
= \sum_{k=1}^{50} 2k - \sum_{k=1}^{50} 1 = 2 \sum_{k=1}^{50} k - \sum_{k=1}^{50} 1 \\
= 2 \left( \frac{50 \cdot 51}{2} \right) - 50 = 2500
\]

Example: Express the sum in closed form

\[
\sum_{k=1}^{n} \frac{3k^2}{n} = 3 \sum_{k=1}^{n} \frac{k^2}{n} = 3 \left( \frac{n(n+1)(2n+1)}{6} \right) \\
= \frac{(n+1)(2n+1)}{2}
\]
Back to the Area Problem:
Strategy (overview):

- Divide the area into $n$ rectangles
- Sum up the areas of the rectangles to get an approximation to the area under the curve
• Increase the number of rectangles to get better approximations.

• To get the exact area, use an "infinite" number of rectangles, i.e. take \( \lim_{n \to +\infty} \)
Strategy (details):

- Divide the interval \([a, b]\) into \(n\) equal subintervals, each with length \(\Delta x = \frac{b-a}{n}\).

These \(n\) subintervals will be the bases of the \(n\) rectangles.
Within the $k^{th}$ subinterval ($k=1, 2, \ldots, n$) pick a value $x_k^*$. Let $f(x_k^*)$ be the height of the $k^{th}$ rectangle.

Note: Common choices for $x_k^*$ are the left endpoint, midpoint, or right endpoint of the $k^{th}$ subinterval.
• Sum up the areas of the \( n \) rectangles

\[
\sum_{k=1}^{n} f(x_k^*) \Delta x
\]

This sum is called a Riemann Sum, and its lower limit of summation will be 1.

• To get the exact area under the curve, use the limit

\[
\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x
\]
The Big Picture:

Given a function $f(x)$ that is nonnegative and continuous on $[a,b]$, the area under the curve and over $[a,b]$ is

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x$$

- $f(x_k^*)$ is the height of the $k^{th}$ rectangle
- $\Delta x$ is the length of base of the $k^{th}$ rectangle
- $\sum_{k=1}^{n} f(x_k^*) \Delta x$ is the approximate area under the curve (Riemann Sum)
- The exact area under the curve
Examples

Find the area under \( f(x) = 6 - x \) and over \([1, 5]\)

1. Use right endpoints for \( x^*_k \)

\[
\Delta x = \frac{b-a}{n} = \frac{5-1}{n} = \frac{4}{n}
\]

length of each subinterval;

length of base of each rectangle

Subintervals: \([1, 1 + \frac{4}{n}], [1 + \frac{4}{n}, 1 + \frac{8}{n}], [1 + \frac{8}{n}, 1 + \frac{12}{n}]\), ...

Right endpoints: \(1 + \frac{4}{n}, \ 1 + \frac{8}{n}, \ 1 + \frac{12}{n}\), ...

Index: \(k = 1, k = 2, k = 3, \ldots\)
So \( X_k^* = 1 + \frac{4k}{n} \) - right endpoint of \( k^{th} \) subinterval

In general, for right endpoints
\[
X_k^* = a + k \Delta x
\]

\[
f(X_k^*) = 6 - (1 + \frac{4k}{n}) = 5 - \frac{4k}{n}
\]

Form the Riemann Sum
\[
\sum_{k=1}^{n} \left( 5 - \frac{4k}{n} \right) \left( \frac{4}{n} \right)
\]

Need this in closed form
\[
= \frac{4}{n} \sum_{k=1}^{n} \left( 5 - \frac{4k}{n} \right) = \frac{4}{n} \left[ \sum_{k=1}^{n} 5 - \sum_{k=1}^{n} \frac{4k}{n} \right]
\]

\[
= \frac{4}{n} \left[ 5 \sum_{k=1}^{n} 1 - \frac{4}{n} \sum_{k=1}^{n} k \right] = \frac{4}{n} \left[ 5n - \frac{4}{n} \frac{n(n+1)}{2} \right]
\]
\[
= \frac{4}{n} \left[ 5n - 2(n+1) \right] = \frac{4}{n} (3n - 2)
\]

= \frac{12n - 8}{n} \quad \text{approximate area using} \quad n \text{ rectangles and right endpoints}

\text{To find exact area, take limit.}

\[
\lim_{n \to +\infty} \frac{12n - 8}{n} = \lim_{n \to +\infty} \frac{12 - \frac{8}{n}}{1} = 12
\]
(2) Use left endpoints

\[ \Delta x = \frac{4}{n} \]

Subintervals: \([1, 1 + \frac{4}{n}], [1 + \frac{4}{n}, 1 + \frac{8}{n}], [1 + \frac{8}{n}, 1 + \frac{12}{n}], \ldots \]
Right endpoints: \(1, 1 + \frac{4}{n}, 1 + \frac{8}{n}, \ldots\)

Index: \(k = 1, 2, 3, \ldots\)

So \(x_k^* = \left(1 + \frac{4(k-1)}{n}\right)\) left endpoint of \(k^{th}\) subinterval

In general, for left endpoints

\[ x_k^* = a + (k-1) \Delta x \]

\[ f(x_k^*) = 6 - \left(1 + \frac{4(k-1)}{n}\right) = 5 - \frac{4(k-1)}{n} \] height of \(k^{th}\) rectangle
\[
\sum_{k=1}^{n} \left( 5 - \frac{4(k-1)}{n} \right) \left( \frac{4}{n} \right) = \frac{4}{n} \sum_{k=1}^{n} \left( 5 - \frac{4(k-1)}{n} \right)
\]

\[
= \frac{4}{n} \left[ \sum_{k=1}^{n} 5 - \sum_{k=1}^{n} \frac{4(k-1)}{n} \right] = \frac{4}{n} \left[ 5 \sum_{k=1}^{n} 1 - \frac{4}{n} \sum_{k=1}^{n} (k-1) \right]
\]

\[
= \frac{4}{n} \left[ 5 \sum_{k=1}^{n} 1 - \frac{4}{n} \left( \sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \right) \right]
\]

\[
= \frac{4}{n} \left[ 5n - \frac{4}{n} \left( \frac{n(n+1)}{2} - n \right) \right] = \frac{4}{n} \left[ 5n - 2n(n+1) + 4 \right]
\]

\[
= \frac{4}{n} \left( 5n - 2n^2 - 2 + 4 \right) = \frac{4}{n} \left( 3n+2 \right)
\]

= \frac{12n + 8}{n} \hspace{1cm} \text{approximate area using } \newline \text{n rectangles and left endpoints}

\text{Exact area: } \lim_{{n \to +\infty}} \frac{12n + 8}{n} = 12
(3) Use midpoints. Exercise.
\[ \lim_{n \to \infty} x^*_k = a + (k - \frac{1}{2}) \Delta x \]

(4) Use geometry

\[
\begin{align*}
\text{area of } \Delta &= \frac{1}{2} (4)(4) = 8 \\
\text{area of } \square &= (4)(1) = 4 \\
\text{Exact area: } &8 + 4 = 12 
\end{align*}
\]

More exercises

Find the area under \( f(x) = x^2 \) and over \([0, 1]\)

(5) Use right endpoints

(6) Use left endpoints

(7) Use midpoints
If we apply the formula
\[
\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x
\]
to a function $f$ that is continuous but not necessarily nonnegative on $[a,b]$, we get the net-signed area between $f(x)$ and $[a,b]$, i.e., the area above the $x$-axis minus the area below the $x$-axis.

\[A_1, A_2, A_3 > 0\]
Net-signed area is
\[A_1 + A_2 - A_3\]
Even more exercises

Find the net-signed area between \( f(x) = 6 - x \) and \([0, 7]\)

8) Use right endpoints
9) Use geometry