Chapter 8.4: First-Order Linear ODE's

Form: \[ \frac{dy}{dx} + p(x) y = q(x) \]

Special Case: \( p(x) = 0 \) \( \Rightarrow \frac{dy}{dx} = q(x) \)

e.g. \( \frac{dy}{dx} = x \sqrt{x^2 + 4} \)

Just integrate both sides wrt \( x \)

\[ y = \int x \sqrt{x^2 + 4} \, dx \]

\[ = \frac{1}{3} (x^2 + 4)^{\frac{3}{2}} + C \quad \text{(Verify)} \]
\[ p(x) \neq 0, \text{ e.g.} \]
\[
\frac{dy}{dx} + 3y = e^{-2x} \quad (\text{i.e. } p(x) = 3, q(x) = e^{-2x})
\]

Multiply both sides by

\[ \mu(x) = e^{3x} \quad (\mu(x) \text{ is called the integrating factor of the ODE}) \]

to get

\[
e^{3x} \frac{dy}{dx} + y(3e^{3x}) = e^{-2x} e^{3x} = e^x
\]

\[ \text{Result of a product rule} \]
\[
\frac{d}{dx} (e^{3x} \cdot y) = e^x
\]
Now integrate both sides wrt $x$.

$$S \frac{d}{dx} (e^{3x} y) \, dx = S e^x \, dx$$

$$e^{3x} y = e^x + C$$

So

$$y = e^x + C \quad \Rightarrow \quad y = e^{-2x} + Ce^{-3x}$$

Check solution:

$$\frac{dy}{dx} + 3y = -2e^{-2x} - 3Ce^{-3x} + 3e^{-2x} + 3Ce^{-3x}$$

$$= e^{-2x} \quad \checkmark$$
How do we compute \( \mu(x) \)?

\[
\mu(x) = e^{\int p(x) \, dx}
\]

e.g. \( p(x) = 3 \Rightarrow e^{\int 3 \, dx} = e^{3x} \quad \text{Don't need a} \ + C \quad \text{at this step}

Why does multiplying \( \frac{dy}{dx} + p(x) y = q(x) \) by \( \mu(x) \) yield a product rule?

If \( \mu(x) = e^{\int p(x) \, dx} \), then

\[
\frac{d}{dx} (\mu y) = e^{\int p(x) \, dx} \frac{d}{dx} \left( \int p(x) \, dx \right)
\]

\[
= e^{\int p(x) \, dx} p(x)
\]

\[
= \mu(x) p(x) = \mu p
\]
So \( \frac{dy}{dx} = \mu p \)

If we multiply \( \frac{dy}{dx} + p(x)y = q(x) \) by \( \mu \) we get

\[
\mu \frac{dy}{dx} + \mu p y = \mu q(x)
\]

\[
\frac{d}{dx} (\mu y) = \mu q(x)
\]

Now integrate both sides wrt \( x \), etc.
Example: Solve the initial-value problem

\[
\begin{align*}
\frac{d}{dx} x^2 \frac{dy}{dx} + xy &= x \sin x, \quad x > 0 \\
y(\pi) &= 2
\end{align*}
\]

\[x^2 \frac{dy}{dx} + xy = x \sin x\]

Divide by \(x^2\) (\(\neq 0\) because \(x > 0\))

\[\frac{dy}{dx} + \frac{1}{x} y = \frac{\sin x}{x}\]

(i.e. \(p(x) = \frac{1}{x}\), \(q(x) = \frac{\sin x}{x}\))

\[
\mu(x) = e^{\int \frac{1}{x} \, dx} = e^{\ln |x|} = e^{\ln x} = x
\]
So \( x \frac{dy}{dx} + y = \sin x \)

\[ \frac{d}{dx} (xy) = \sin x \]

\[ \int \frac{d}{dx} (xy) \, dx = \int \sin x \, dx \]

\( xy = -\cos x + C \)

\[ y = \frac{-\cos x + C}{x} \]

\( y(\pi) = 2 \quad \Rightarrow \quad 2 = -\frac{\cos \pi}{\pi} + \frac{C}{\pi} \quad \Rightarrow \quad 2\pi = 1 + C \]

\[ C = 2\pi - 1 \]

So \( y = \frac{-\cos x + 2\pi - 1}{x} \)
Mixing Problems

Idea: A tank is filled to some level with a solution that contains a soluble substance. The solution is added to the tank and drained from the tank at various rates.

Goal: Find the amount of the substance in the tank at any time $t$.

Example: A tank initially contains 50 pounds of salt dissolved in 500 gallons of water. Brine containing 5 pounds of salt per gallon is added to the tank at a rate of 20 gallons per minute and is drained at the same rate. How much salt is in the tank at an arbitrary time $t$?

Let $y = y(t)$ be the amount of salt (in pounds) at time $t$. 
\[
\frac{dy}{dt} = \text{rate in} - \text{rate out}
\]

**rate in:** \(5 \text{ lb} \cdot \text{gal}^{-1}\) \(\frac{\text{lb}}{\text{gal}}\), \(20 \text{ gal} \cdot \text{min}^{-1}\) \(\frac{\text{gal}}{\text{min}}\) = \(100 \text{ lb} \cdot \text{min}^{-1}\) \(\frac{\text{lb}}{\text{min}}\)

**concentration of substance in solution entering**

**flow rate of solution entering**

**rate out:** \(y(\text{lb}) \cdot \frac{\text{lb}}{500 \text{ gal}}\), \(20 \text{ gal} \cdot \text{min}^{-1}\) \(\frac{\text{gal}}{\text{min}}\) = \(\frac{1}{25}y \cdot \frac{\text{lb}}{\text{min}}\)

**concentration of substance in solution exiting**

**flow rate of solution exiting**

**Note:** If flow rate entering and exiting are not the same, this volume would be a function of \(t\) instead of a constant.
So we must solve initial-value problem
\[
\begin{cases}
\frac{dy}{dt} = 100 - \frac{1}{25} y \\
y(0) = 50
\end{cases}
\]
\[
\frac{dy}{dt} = 100 - \frac{1}{25} y \quad \Rightarrow \quad \frac{dy}{dt} + \frac{1}{25} y = 100
\]

Use integrating factors.
\[
\text{mult} = e^{\int \frac{1}{25} dt} = e^{\frac{t}{25}}
\]
\[
e^{\frac{t}{25}} \frac{dy}{dt} + \frac{1}{25} e^{\frac{t}{25}} y = 100 e^{\frac{t}{25}}
\]
\[
\frac{d}{dt} \left( e^{\frac{t}{25}} y \right) = 100 e^{\frac{t}{25}}
\]
\[
\int \frac{d}{dt} \left( e^{\frac{t}{25}} y \right) dt = \int 100 e^{\frac{t}{25}} dt
\]
\[ e^{\frac{t}{25}} y = 100 \cdot 25 - e^{\frac{t}{25}} + C = 2500 e^{\frac{t}{25}} + C \]

\[ y(t) = 2500 + C e^{-\frac{t}{25}} \]

\[ y(0) = 50 \quad \text{so} \]

\[ 50 = 2500 + C e^0 \]

\[ C = -2450 \]

Solution: \[ y(t) = 2500 - 2450 e^{-\frac{t}{25}} \]
We could also use separation of variables:

\[ \frac{dy}{dt} = 100 - \frac{1}{25} y \quad \Rightarrow \quad 25 \frac{dy}{dt} = 2500 - y \]

\[ \Rightarrow \quad \frac{dy}{2500 - y} = \frac{1}{25} \quad dt \]

\[ \int \frac{dy}{2500 - y} = \int \frac{1}{25} \quad dt \]

\[ -\ln |2500 - y| = \frac{1}{25} t + C \quad \Rightarrow \quad \ln |2500 - y| = -\frac{1}{25} t - C \]

\[ |2500 - y| = e^{-\frac{1}{25} t - C} = e^{-C} e^{-\frac{t}{25}} \]

\[ 2500 - y = \pm e^{-C} e^{-\frac{t}{25}} = K e^{-\frac{t}{25}} \]

\[ y(t) = 2500 - Ke^{-\frac{t}{25}} \]

\[ y(0) = 50 \quad \Rightarrow \quad 50 = 2500 - K \quad \Rightarrow \quad K = 2450 \]

Solution: \[ y(t) = 2500 - 2450 e^{-\frac{t}{25}} \]
Note: In the previous example if the flow rate out of the solution was different than the flow rate in,

e.g. \( 5 \text{ gal/min} \), what would change?

Rate out: \( \frac{y(t)}{500 + (20 - 5)t} \text{ gal} \cdot 5 \text{ gal/min} = \frac{5y}{500 + 15t} \text{ lb/min} \)

\[ = \frac{y}{100 + 3t} \text{ lb/min} \]

So we would have

\[ \begin{cases} \frac{dy}{dt} = 100 - \frac{y}{100 + 3t} \\ y(0) = 50 \end{cases} \]