Chapter 10.7: Polynomial Approximation of Functions
(Maclaurin and Taylor Polynomials)

Idea: Approximate "complicated" functions (e.g., $e^x$, $\cos x$) with "simpler" functions (e.g., polynomials)

Motivating Example: Approximate $e^{0.1}$

Recall: The local linear approximation of $f(x)$ at $x = x_0$ is

$$p(x) = f(x_0) + f'(x_0)(x-x_0)$$

Note: This is just the equation of the tangent line to $f(x)$ at $x = x_0$
e.g. \[ f(x) = e^x \text{, } x_0 = 0 \]

\[ f'(x) = e^x \]

\[ f'(0) = e^0 = 1 \text{ is the slope of the tangent line to } f(x) = e^x \text{ at } x = 0 \]

\[ y - y_0 = m(x - x_0) \]

\[ y_0 = f(0) = e^0 = 1 \]

\[ y - 1 = 1(x - 0) \]

\[ y = 1 + x \]

So \[ p_1(x) = 1 + x \text{ is the local linear approximation to } f(x) = e^x \text{ at } x = 0. \]
So "near" $x = 0$

$e^x \approx 1 = x$

because $e^0 = 1 + 0$

and $\frac{d}{dx} (e^x)$ at $x = 0 = \frac{d}{dx} (1 + x)$ at $x = 0$

i.e.

$p(0)$ and $f(0)$ go through the same point and have the same slope at $x = 0$

So $e^{0.1} \approx 1 + 0.1 = 1.1$

Calculator: $e^{0.1} \approx 1.105170918$
Graph:

\[ f(x) = e^x \]

\[ p_1(x) = 1 + x \]

With a 2nd-degree polynomial we can do better, i.e. find a quadratic function \( p_2(x) \) that has the same point, same slope, and same concavity as \( f(x) = e^x \) at \( x = 0 \).
\( p_2(x) = c_0 + c_1 x + c_2 x^2 \)
\( p_2(0) = c_0 \)
\( p_2'(x) = c_1 + 2c_2 x \)
\( p_2'(0) = c_1 \)
\( p_2''(x) = 2c_2 \)
\( p_2''(0) = 2c_2 \)

\( f(x) = e^x \)
\( f(0) = e^0 = 1 \)
\( f'(x) = e^x \)
\( f'(0) = e^0 = 1 \)
\( f''(x) = e^x \)
\( f''(0) = e^0 = 1 \)

Want \( f(0) = p_2(0) \Rightarrow c_0 = 1 \)
\( f'(0) = p_2'(0) \Rightarrow c_1 = 1 \)
\( f''(0) = p_2''(0) \Rightarrow 2c_2 = 1 \Rightarrow c_2 = \frac{1}{2} \)
So

\[ p_2(x) = 1 + x + \frac{1}{2} x^2 \]

is the local quadratic approximation to \( f(x) = e^x \) at \( x = 0 \)

So \( e^{0.1} \approx 1 + 0.1 + \frac{1}{2} (0.1)^2 \)

\[ = 1 + 0.1 + 0.005 = 1.105 \]

Calculator: \( e^{0.1} \approx 1.105470918 \)
Example: Use a local quadratic approximation to approximate \( \cos 2^\circ \).

\[
f(x) = \cos x, \quad x_0 = 0
\]

Recall:

\[
p_2(x) = c_0 + c_1 x + c_2 x^2
\]
\[
p_2(0) = c_0
\]
\[
p_2'(0) = c_1
\]
\[
p_2''(0) = 2c_2
\]

Now

\[
f(x) = \cos x
\]
\[
f(0) = \cos 0 = 1 \quad \Longrightarrow \quad c_0 = 1
\]
\[
f'(x) = -\sin x \quad \Longrightarrow \quad f'(0) = 0 \quad \Longrightarrow \quad c_1 = 0
\]
\[
f''(x) = -\cos x \quad \Longrightarrow \quad f''(0) = -1
\]
\[
\Longrightarrow 2c_2 = -1 \quad \Longrightarrow \quad c_2 = -\frac{1}{2}
\]
So \( p_2(x) = 1 - \frac{1}{2}x^2 \) is the local quadratic approximation of \( f(x) = \cos x \) at \( x = 0 \).

So \( \cos 2^\circ = \cos \left( \frac{\pi}{90} \right) \)

\[ \approx 1 - \frac{1}{2} \left( \frac{\pi}{90} \right)^2 \]

\[ \approx 0.999390765 \]

Calculator: \( \cos 2^\circ \approx 0.999390827 \)
With higher degree polynomials, we can get more accurate approximations.

Goal: Assume $f(x)$ can be differentiated $n$ times at $x = 0$. Find a polynomial $p_n(x)$ such that

\[ p_n(0) = f(0) \]
\[ p_n'(0) = f'(0) \]
\[ p_n''(0) = f''(0) \]
\[ \vdots \]
\[ p_n^{(n)}(0) = f^{(n)}(0) \]

[Notation: $(n)$ superscript means "$n^{th}$ derivative"]
\[ p_n(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n \]
\[ p_n(0) = c_0 = f(0) \]
\[ p_n'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \ldots + nc_n x^{n-1} \]
\[ p_n'(0) = c_1 = f'(0) \]
\[ p_n''(x) = 2c_2 + 3 \cdot 2c_3 x + \ldots + n(n-1)c_n x^{n-2} \]
\[ p_n''(0) = 2c_2 = f''(0) \]
\[ c_2 = \frac{1}{2} f''(0) = \frac{1}{2} f''(0) \]
\[ p_n'''(x) = 3 \cdot 2c_3 + \ldots + n(n-1)(n-2)c_n x^{n-3} \]
\[ p_n'''(0) = 3 \cdot 2c_3 = f'''(0) \]
\[ c_3 = \frac{1}{3 \cdot 2} f'''(0) = \frac{1}{3 \cdot 2} f'''(0) \]
\[ \vdots \]
\[ p_n^{(n)}(x) = n(n-1)(n-2)\ldots 1 \cdot c_n = n! \cdot c_n \]
\[ p_n^{(n)}(0) = n! \cdot c_n = f^{(n)}(0) \]
\[ c_n = \frac{1}{n!} f^{(n)}(0) \]

So
\[ p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots + \frac{f^{(n)}(0)}{n!} x^n \]

or
\[ p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \]

Convention:
\[ f^{(0)}(0) = f(0) \]

This is called the \( n \)th Maclaurin polynomial for \( f(x) \)
Note: $p_1(x)$: local linear approximation of $f(x)$ at $x=0$
$p_2(x)$: local quadratic approximation of $f(x)$ at $x=0$

Examples

1) Find Maclaurin polynomials $p_3, p_4,$ and $p_n$

for $f(x) = e^x$

$f(x) = f'(x) = f''(x) = \ldots = f^{(n)}(x) = e^x$

$f(0) = f'(0) = f''(0) = \ldots = f^{(n)}(0) = e^0 = 1$

So

$p_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$p_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$p_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{1}{k!} x^k$
(2) Find Maclaurin polynomial $p_n$ for $f(x) = \cos x$

$f'(x) = -\sin x \Rightarrow f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = \sin 0 = 0$

\[ \begin{align*}
\ddots & \\
\text{repeats} & \\

p_0(x) &= 1, \quad p_1(x) = 1 + 0x = 1 \\
p_0(x) &= p_1(x) \\
p_2(x) &= 1 - \frac{x^2}{2!} = p_3(x) \\
p_4(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} = p_5(x) \\
&\vdots
\end{align*} \]
So for $k = 0, 1, 2, \ldots$

$$p_{2k}(x) = p_{2k+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^k}{(2k)!} x^{2k}$$

We can write

$$p_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k}{(2k)!} x^{2k}$$

where $\left\lfloor \frac{n}{2} \right\rfloor$ is the greatest integer $\leq \frac{n}{2}$

e.g. $n = 4 \Rightarrow \left\lfloor \frac{4}{2} \right\rfloor = 2 \Rightarrow p_4(x) = p_5(x)$

$n = 5 \Rightarrow \left\lfloor \frac{5}{2} \right\rfloor = 2 \Rightarrow p_4(x) = p_5(x)$
To approximate \( f(x) \) near an arbitrary \( x = x_0 \),
we use the polynomial
\[
p_n(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \ldots + c_n(x-x_0)^n
\]
Now \( p_n(x_0) = c_0 = f(x_0) \)
and it can be shown that
\[
c_1 = f'(x_0)
\]
\[
c_2 = \frac{f''(x_0)}{2!}
\]
\[
c_3 = \frac{f'''(x_0)}{3!}
\]
\[ \vdots \]
\[
c_n = \frac{f^{(n)}(x_0)}{n!}
\]
So
\[ p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \]

or
\[ p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \]

This is called the \( n^{\text{th}} \) Taylor polynomial for \( f(x) \) about \( x = x_0 \).
Examples

1) Find the Taylor polynomial of degree 4 for
\[ f(x) = \sin x \] about \( x = \frac{\pi}{2} \)

\[ f\left(\frac{\pi}{2}\right) = 1 \]
\[ f'\left(\frac{\pi}{2}\right) = 0 \]
\[ f''\left(\frac{\pi}{2}\right) = -1 \]
\[ f'''\left(\frac{\pi}{2}\right) = 0 \]
\[ f^{(4)}\left(\frac{\pi}{2}\right) = 1 \]

So
\[ p_4(x) = 1 - \frac{1}{2!} (x-\frac{\pi}{2})^2 + \frac{1}{4!} (x-\frac{\pi}{2})^4 \]
(2) Find the $n^{th}$ Taylor polynomial for $f(x) = \frac{1}{x}$ about $x = 1$

\[ f(x) = \frac{1}{x} \implies f(1) = 1 \]

\[ f'(x) = -\frac{1}{x^2} \implies f'(1) = -1 \]

\[ f''(x) = \frac{2}{x^3} \implies f''(1) = 2 = 2! \]

\[ f'''(x) = -\frac{6}{x^4} \implies f'''(1) = -6 = -3! \]

\[ f^{(4)}(x) = \frac{24}{x^5} \implies f^{(4)}(1) = 4! \]

\[ \vdots \]

\[ f^{(n)}(x) = (\xi-1)^n n! \implies f^{(n)}(1) = (\xi-1)^n \cdot n! \]
So

\[ p_n(x) = 1 - (x-1) + \frac{2!}{2!} (x-1)^2 - \frac{3!}{3!} (x-1)^3 + \frac{4!}{4!} (x-1)^4 \]

\[ + \cdots + (x-1)^n n! \frac{n!}{n!} (x-1)^n \]

or \[ p_n(x) = \sum_{k=0}^{n} (-1)^k (x-1)^k \]