Chapter 9.1: Intro to Differential Equations

Definition: An ordinary differential equation (ODE) is an equation that includes the derivative of some unknown function. The order of an ODE is the order of the highest order derivative in the equation, e.g.

\[ \frac{dy}{dx} = x \sqrt{x^2 + y} \]

1st order

\[ xy' = (1-x)y \]

1st order

\[ \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0 \]

2nd order

\[ m \frac{d^2y}{dt^2} = -mg \]

2nd order

free fall model

\[ \frac{d^2y}{dt^2} + \frac{c}{m} \frac{dy}{dt} = -g \]

2nd order

free fall subject to air resistance
The solution to an ODE is a function that satisfies the equation.

Example: Verify that \( y = Ae^{2x} + Be^{-4x} \) (\( A, B \) constants) is a solution to the ODE \( \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0 \).

If \( y = Ae^{2x} + Be^{-4x} \)

\[ \Rightarrow \frac{dy}{dx} = 2Ae^{2x} - 4Be^{-4x} \text{ and } \frac{d^2y}{dx^2} = 4Ae^{2x} + 16Be^{-4x} \]

So \( \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y \)

\[ = 4Ae^{2x} + 16Be^{-4x} + 2(2Ae^{2x} - 4Be^{-4x}) - 8(Ae^{2x} + Be^{-4x}) \]

\[ = 4Ae^{2x} + 16Be^{-4x} + 4Ae^{2x} - 8Be^{-4x} - 8Ae^{2x} - 8Be^{-4x} \]

\[ = 8Ae^{2x} - 8Ae^{2x} + 16Be^{-4x} - 16Be^{-4x} \]

\[ = 0 \quad \checkmark \]
Practice: Verify that \( y = xe^{-x} \) is a solution
to \( xy' = (1-x)y \)

\textbf{Solving ODE's}

(1) First-Order Linear Equations

Form: \( \frac{dy}{dx} + p(x)y = q(x) \)

(a) Special Case: \( p(x) = 0 \), i.e. \( \frac{dy}{dx} = q(x) \)

E.g. to solve \( \frac{dy}{dx} = x \sqrt{x^2+4} \)

we just integrate both sides with respect to (wrt) \( x \)

\[ y = \int x \sqrt{x^2+4} \, dx \quad u=\text{sub:} \ u=x^2+4 \]

\[ y = \frac{1}{3} (x^2+4)^{3/2} + C \quad (\text{show details}) \]
Recall how to solve initial-value problems:

E.g., \[
\begin{align*}
\frac{dy}{dx} &= x\sqrt{x^2+y} \\
y(-4) &= 0
\end{align*}
\]

\[
y = \frac{1}{3} (x^2+y)^{\frac{3}{2}} + C
\]

\[
0 = \frac{1}{3} (-4)^{\frac{3}{2}} + C = \frac{1}{3} (-20)^{\frac{3}{2}} + C
\]

\[
C = \frac{-40\sqrt{5}}{3}
\]

Solution: \[
y = \frac{1}{3} (x^2+y)^{\frac{3}{2}} - \frac{40\sqrt{5}}{3}
\]
(b) \( p(x) \neq 0 \)

E.g. \( \frac{dy}{dx} + 3y = e^{-2x} \), i.e. \( p(x) = 3, q(x) = e^{-2x} \)

Multiply both sides by \( \mu(x) = e^{3x} \) (Later we'll see where this comes from)

\[
e^{3x} \frac{dy}{dx} + y(3e^{3x}) = e^{-2x}e^{3x} = e^x
\]

\[
\text{Result of a product rule}
\]

\[
\frac{d}{dx}(e^{3x}y) = e^x
\]

Now integrate both sides w.r.t. \( x \)

\[
\int \frac{d}{dx}(e^{3x}y) \, dx = \int e^x \, dx
\]
\[ e^{3x}y = e^x + C \quad \text{Note: We only need one } + C, \text{ usually on RHS} \]

\[ y = \frac{e^x}{e^{3x}} + \frac{C}{e^{3x}} = e^{-2x} + Ce^{-3x} \]

Check solution:

\[ \frac{dy}{dx} + 3y = e^{-2x} \]

\[ y = e^{-2x} + Ce^{-3x} \implies \frac{dy}{dx} = -2e^{-2x} - 3Ce^{-3x} \]

\[ \frac{dy}{dx} + 3y = -2e^{-2x} - 3Ce^{-3x} + 3(e^{-2x} + Ce^{-3x}) \]

\[ = -2Ce^{-2x} - 3Ce^{-3x} + 3e^{-2x} + 3Ce^{-3x} \]

\[ = e^{-2x} \quad \checkmark \]
Now where did \( \mu(x) = e^{3x} \) come from?

We always let \( \mu(x) = e^{\int p(x) \, dx} \), so

if \( p(x) = 3 \) then \( \mu(x) = e^{\int 3 \, dx} = e^{3x} \)

Note: Don't need +C here since we can choose any antiderivative of \( p(x) \).

Claim: If we multiply both sides of \( \frac{dy}{dx} + p(x)y = q(x) \) by 

\( \mu(x) = e^{\int p(x) \, dx} \) the LHS will be the result of a product rule, namely \( \frac{d}{dx}(\mu y) \).

Here's why: Let \( \mu(x) = e^{\int p(x) \, dx} \)

Note that \( \frac{d}{dx} = e^{\int p(x) \, dx} \cdot \frac{d}{dx} (e^{\int p(x) \, dx}) = e^{\int p(x) \, dx} p(x) \)

\( \frac{dy}{dx} = \mu p(x) \)

Now multiply \( \frac{dy}{dx} + p(x)y = q(x) \) by \( \mu(x) \) to get

\( \mu(x) \frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x) \)
\[ \mu \frac{dy}{dx} + \mu p(x)y = \mu q(x) \]
\[ \mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu q(x) \]
\[ \text{Result of product rule} \]
\[ \frac{d}{dx} (\mu y) = \mu q(x) \]

Now we would integrate both sides with respect to \( x \) and solve for \( y \).

**Example:** Solve the initial-value problem

\[
\begin{cases}
  x^2 \frac{dy}{dx} + xy = x\sin x, & x > 0 \\
  y(\pi) = 2
\end{cases}
\]
\[ x^2 \frac{dy}{dx} + xy = x \sin x \]

Divide by \( x^2 \) (\( \neq 0 \) because \( x > 0 \))

\[ \frac{dy}{dx} + \frac{1}{x} y = \frac{\sin x}{x} \]

\( p(x) = \frac{1}{x} \), \( q(x) = \frac{\sin x}{x} \)

\( \mu(x) = e^{\int \frac{1}{x} \, dx} = e^{\ln|x|} = e^{\ln x} = x \)

So \( x \frac{dy}{dx} + y = \sin x \)

\[ \frac{d}{dx} (xy) = \sin x \]

\( \int \frac{d}{dx} (xy) \, dx = \int \sin x \, dx \)

\[ xy = -\cos x + C \]
\[ y = \frac{-\cos x}{x} + \frac{C}{x} \]

Now \[ y(\pi) = \frac{1}{2} \]

\[ 2 = \frac{-\cos \pi}{\pi} + \frac{C}{\pi} = -\frac{-1}{\pi} + \frac{C}{\pi} = \frac{1+C}{\pi} \]

\[ 1+C = 2\pi \quad \Rightarrow \quad C = 2\pi - 1 \]

Solution: \[ y = \frac{-\cos x}{x} + \frac{2\pi-1}{x} \]
(2) First-Order Separable Equations

Form: \( h(y) \frac{dy}{dx} = g(x) \)

We can "separate" the variables to either side

\[ h(y) \, dy = g(x) \, dx \]

Differential form

Now integrate both sides (LHS wrt y, RHS wrt x)

\[ \int h(y) \, dy = \int g(x) \, dx \]

where \( \frac{d}{dy} h(y) = h(y) \)

\( G'(x) = g(x) \)

This is an implicit solution of the ODE. If we can, we should solve for \( y \) explicitly.
Example

Solve: \( \frac{dy}{dx} = 2(1+y^2) x \)

\( \frac{dy}{1+y^2} = 2x \, dx \quad \text{Note: } 1+y^2 \neq 0 \)

\[ \int \frac{dy}{1+y^2} = \int 2x \, dx \]

\[ \tan^{-1} y = x^2 + C \]

\[ y = \tan(x^2 + C) \]

Check: \( y = \tan(x^2 + C) \implies \)

\( \frac{dy}{dx} = \sec^2(x^2 + C) \cdot \frac{d}{dx}(x^2 + C) \Rightarrow \frac{dy}{dx} = 2x \sec^2(x^2 + C) \)

RHS of ODE: \( 2x(1+y^2) = 2x\left(1 + \tan^2(x^2 + C)\right) = 2x \sec^2(x^2 + C) \)

same \( \checkmark \)
Example: Solve \((4y - \cos y) \frac{dy}{dx} = 3x^2\)

\((4y - \cos y) \, dy = 3x^2 \, dx\)

\(\int (4y - \cos y) \, dy = \int 3x^2 \, dx\)

\(2y^2 - \sin y = x^3 + C\)

We can't write \(y\) as an explicit function of \(x\), so we leave the solution in implicit form.
Example: Solve \( \frac{dy}{dx} = -xy \)

If \( y \neq 0 \):
\[
\int \frac{dy}{y} = \int -x \, dx
\]

\[
\ln |y| = -\frac{1}{2}x^2 + C
\]

\[
|y| = e^{-\frac{1}{2}x^2 + C} = e^C e^{-\frac{1}{2}x^2}
\]

\[
y = \pm e^C e^{-\frac{1}{2}x^2}
\]

\[
y = K e^{-\frac{1}{2}x^2}, \quad K \neq 0
\]

Solution: \( y = K e^{-\frac{1}{2}x^2} \) (\( K \neq 0 \)) and \( y = 0 \)

(Recommended) \( y = Ke^{-\frac{1}{2}x^2} \) i.e. we can merge the two solutions if we allow \( K = 0 \)

Note: \( y = 0 \) is a solution to the ODE so we must include it in our answer.
Example: Solve $y' + y^2 \sin x = 0$

\[
\frac{dy}{dx} + y^2 \sin x = 0
\]

\[
\frac{dy}{dx} = -y^2 \sin x
\]

If $y \neq 0$:

\[
\frac{dy}{y^2} = -\sin x \, dx
\]

\[
\int \frac{dy}{y^2} = \int -\sin x \, dx
\]

\[
-\frac{1}{y} = \cos x + C
\]

\[
y = -\frac{1}{\cos x + C}
\]

Solutions: $y = -\frac{1}{\cos x + C}$ and $y = 0$

Note: There is no way to merge the solutions in this problem

Note: $y = 0$ is a solution
Example: Find a curve in the xy-plane that passes through (0,3) and whose tangent line at any point \((x,y)\) has slope \(\frac{2x}{y^2}\).

So we need to solve the initial-value problem

\[
\begin{cases}
\frac{dy}{dx} = \frac{2x}{y^2} \\
y(0) = 3
\end{cases}
\]

\[
\frac{dy}{dx} = \frac{2x}{y^2}
\]

\[y^2 \, dy = 2x \, dx\]

\[\int y^2 \, dy = \int 2x \, dx\]

\[\frac{1}{3} y^3 = x^2 + C\]

Now \( y(0) = 3 \)

so \( \frac{1}{3} (3)^3 = 0^2 + C \Rightarrow C = 9 \)

\[ \frac{1}{3} y^3 = x^2 + 9 \]

\[ y^3 = 3x^2 + 27 \]

\[ y = \sqrt[3]{3x^2 + 27} \]
Mixing Problems

Idea: A tank is filled to some level with a solution that contains a soluble substance. The solution is added to the tank and drained from the tank at various rates.

Goal: Find the amount of the substance in the tank at any time $t$.

Example: A tank initially contains 50 pounds of salt dissolved in 500 gallons of water. Brine containing 5 pounds of salt per gallon is added to the tank at a rate of 20 gallons per minute and is drained at the same rate. How much salt is in the tank at an arbitrary time $t$?

Let $y = y(t)$ be the amount of salt (in pounds) at time $t$. 
\[
\frac{dy}{dt} = \text{rate in} - \text{rate out}
\]

rate in: \(\frac{5 \text{ lb}}{\text{gal}}\) \quad \frac{20 \text{ gal}}{\text{min}} = \frac{100 \text{ lb}}{\text{min}}

\underline{\text{concentration of substance in solution entering}}

rate out: \(\frac{y(10)}{500 \text{ gal}}\) \quad \frac{20 \text{ gal}}{\text{min}} = \frac{1}{25} y \frac{\text{lb}}{\text{min}}

\underline{\text{concentration of substance in solution exiting}}

\underline{\text{flow rate of solution entering}}

\underline{\text{flow rate of solution exiting}}

Note: If flow rate entering and exiting are not the same, this volume would be a function of \(t\) instead of a constant.
So we must solve initial-value problem

\[
\begin{cases}
\frac{dy}{dt} = 100 - \frac{1}{25} y \\
y(0) = 50
\end{cases}
\]

\[
\frac{dy}{dt} = 100 - \frac{1}{25} y \quad \Rightarrow \quad \frac{dy}{dt} + \frac{1}{25} y = 100
\]

Use integrating factors.

\[
\mu(t) = e^{\int \mu(t) dt} = e^{\int \frac{1}{25} dt} = e^{\frac{t}{25}}
\]

\[
e^{\frac{t}{25}} \frac{dy}{dt} + \frac{1}{25} e^{\frac{t}{25}} y = 100 e^{\frac{t}{25}}
\]

\[
\frac{d}{dt} \left( e^{\frac{t}{25}} y \right) = 100 e^{\frac{t}{25}}
\]

\[
\int \frac{d}{dt} \left( e^{\frac{t}{25}} y \right) dt = \int 100 e^{\frac{t}{25}} dt
\]
\[ e^{\frac{t}{25}} y = 100 \cdot 25 e^{\frac{t}{25}} + C = 2500 e^{\frac{t}{25}} + C \]

\[ y(t) = 2500 + C e^{\frac{-t}{25}} \]

\[ y(0) = 50 \quad \text{so} \]

\[ 50 = 2500 + C e^0 \]

\[ C = -2450 \]

**Solution:** \[ y(t) = 2500 - 2450 e^{\frac{-t}{25}} \]
We could also use separation of variables:

\[
\frac{dy}{dt} = 100 - \frac{1}{25}y \quad \Rightarrow \quad 25\frac{dy}{dt} = 2500 - y
\]

\[
\Rightarrow \quad \frac{dy}{2500-y} = \frac{1}{25} \, dt
\]

\[
\int \frac{dy}{2500-y} = \int \frac{1}{25} \, dt
\]

\[-\ln|2500-y| = \frac{1}{25}t + C \quad \Rightarrow \quad \ln|2500-y| = -\frac{1}{25}t - C
\]

\[
2500 - y = e^{-\frac{1}{25}t - C} = e^{-c} e^{-\frac{t}{25}}
\]

\[
2500 - y = \pm e^{-c} e^{-\frac{t}{25}} = K e^{-\frac{t}{25}}
\]

\[
y(t) = 2500 - Ke^{-\frac{t}{25}}
\]

\[
y(0) = 50 \quad \Rightarrow \quad 50 = 2500 - K \quad \Rightarrow \quad K = 2450
\]

Solution: \( y(t) = 2500 - 2450e^{-\frac{t}{25}} \)