Chapter 9.3: Modeling With Differential Equations

(1) **Exponential Growth Model**

- models simple population growth where the rate of growth is proportional to the size of the population, i.e., the larger the population, the more rapidly it grows.

Let \( y = y(t) \) be the population at time \( t \)

\( y(0) = y_0 \) be the initial population

\[
\begin{cases}
\frac{dy}{dt} = ky, \quad k > 0 \\
y(0) = y_0
\end{cases}
\]

\( k \) is a constant of proportionality called the growth constant.

Solve this initial-value problem by separation of variables.
\[
\frac{dy}{dt} = ky
\]
\[
\frac{dy}{y} = k\, dt
\]

We assume \( y > 0 \)

(or else we have a boring population)

\[
\int \frac{dy}{y} = \int k\, dt
\]

\[
\ln y = kt + C
\]

\[
y = e^{kt+C} = e^{kt}e^C = Ce^{kt}
\]

Now \( y(0) = y_0 \) so

\[
y_0 = Ce^0 = C \quad \Rightarrow \quad C = y_0
\]

Solution: \( y(t) = y_0e^{kt} \)
Example: An E-coli cell divides into 2 cells every 20 minutes. Let \( y = y(t) \) be the number of cells after \( t \) minutes.

(a) Find formula for \( y(t) \).

\[ y(t) = y_0 e^{kt} \]  \( y_0 = 1 \)

\[ y = e^{kt} \]

Given \( y(20) = 2 \) so

\[ 2 = e^{20k} \]

\[ \ln 2 = \ln e^{20k} = 20k \]

\[ k = \frac{\ln 2}{20} \]

So

\[ y(t) = e^{\frac{\ln 2}{20} t} = (e^{\ln 2})^{\frac{t}{20}} = 2^{\frac{t}{20}} \]
(b) How many cells will there be after 2 hours?

\[ 2 \text{ hours} = \frac{120}{60} = 2^6 = 64 \text{ cells} \]

(c) How long before there are 1,000,000 cells?

\[ 1,000,000 = 2^{\frac{t}{20}} \]

\[ \ln 1,000,000 = \ln 2^{\frac{t}{20}} = \frac{t}{20} \ln 2 \]

So \[ t = 20 \frac{\ln 1,000,000}{\ln 2} \approx 399 \text{ minutes} \]

\[ \text{Careful, there is no law of logarithms to simplify this.} \]
(2) **Logistic Model**

- a more realistic population model where the population eventually levels off to the **carrying capacity** $L$ of the system.

\[
\begin{aligned}
\frac{dy}{dt} &= k\left(1 - \frac{y}{L}\right)y, \quad k > 0 \\
y(0) &= y_0
\end{aligned}
\]

**Note:** If $\frac{y}{L}$ is small, $\frac{dy}{dt} \approx ky$ (like exponential growth model).

- If $y = L$, $\frac{dy}{dt} = 0$
- If $y > L$, $\frac{dy}{dt} < 0$ (**the population has grown too large and thus decreases**)

**Practice:** Solve the initial-value problem. **Hint:** Use partial fractions.

**Solution:**

$$y(t) = \frac{y_0 L}{y_0 + (L - y_0) e^{-kt}}$$

**Note:** As $t \to \infty$, $y \to L$
(3) **Exponential Decay Model**

- Rate of decay is proportional to the amount of the substance present

\[ y = y(t) \]: amount of substance that remains at time \( t \)

\[ y(0) = y_0 \]: initial amount of substance

\[
\begin{cases}
\frac{dy}{dt} = -ky, \quad k > 0 \\
y(0) = y_0
\end{cases}
\]

- \( k \) is the decay constant

**Practice:** Show that the solution to this model is

\[ y(t) = y_0 e^{-kt} \]

**Question:** How long before half of the original substance remains?
If half remains, we have \( y = \frac{1}{2} y_0 \)

So \( y = y_0 e^{-kt} \) becomes

\[
\frac{1}{2} y_0 = y_0 e^{-kt}
\]

\[
\frac{1}{2} = e^{-kt}
\]

\[
\ln \frac{1}{2} = \ln e^{-kt} = -kt
\]

\[
t = \frac{\ln \frac{1}{2}}{-k} = -\frac{\ln \frac{1}{2}}{k} = \frac{\ln \left(\frac{1}{2}\right)^{-1}}{k} = \frac{\ln 2}{k}
\]

So \( t = \frac{\ln 2}{k} \) is the half-life of the substance

Note this does not depend on \( y_0 \)

Exercise: Prove that in the exponential growth model, \( t = \frac{\ln 2}{k} \) is the amount of time it takes for the population to double. This is called the **doubling time** for the population.
Example: Suppose 30% of a radioactive substance decays in 5 years. Find the half-life of the substance.

30% decay $\iff$ 70% remains

$y = y_0 e^{-kt}$

Given $y(5) = 0.7 y_0$

So $0.7 y_0 = y_0 e^{-5k}$

$0.7 = e^{-5k}$

$\ln 0.7 = -5k$

$k = \frac{\ln 0.7}{-5}$

So half-life is $\frac{\ln 2}{\ln 0.7} = \frac{-5 \ln 2}{\ln 0.7} \approx 9.7$ years
Since $t = \frac{\ln 2}{k}$ is the half-life, we can solve for the decay constant $k = \frac{\ln 2}{t}$, e.g. the half-life of carbon-14 is 5730 years, so $k = \frac{\ln 2}{5730} \approx 0.000121$ is the decay constant of carbon-14.

**Example:** Shroud of Turin (pg 607)

Test done in 1988, fibers in cloth contained 93% of the original carbon-14. Determine year of origin of cloth.

$$Y = Y_0 e^{-kt} = Y_0 e^{-0.000121t}$$

We want $t$ when $\frac{Y}{Y_0} = 0.93$

$$0.93 = e^{-0.000121t} \Rightarrow \ln 0.93 = -0.000121t$$

$t \approx 600$ years $\Rightarrow$ Year of origin $\approx 1988 - 600 = 1388$
(4) **Newton's Law of Cooling**

Fact: The rate at which a temperature of a cooling object decreases (or a warming object increases) is proportional to the difference between the temperature of the object and the temperature of the surrounding medium.

**Example** Set up an initial value problem for this model.

Let \( T(t) \) be the temperature of the object at time \( t \)

\( T(0) = T_o \) be the initial temperature of the object

\( T_m \) : constant temperature of surrounding medium

\[
\begin{align*}
\frac{dT}{dt} &= k (T_m - T), \quad k > 0 \\
T(0) &= T_o
\end{align*}
\]

Your homework problems ask you to solve this so you can answer questions about specific scenarios.