Chapter 12.1: Vector-valued functions (Curves in space)

Recall: The parametric equations of a line through $P_0 : (x_0, y_0, z_0)$ and parallel to $\vec{v} = (a, b, c)$ are

\[
\begin{align*}
X &= x_0 + at \\
Y &= y_0 + bt \\
Z &= z_0 + ct
\end{align*}
\]

Note that $x, y, z$ are functions of $t$.

The vector equation of the line is

\[
\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle
\]

or

\[
\vec{r}(t) = \vec{r}_0 + t \vec{v}
\]

$\vec{r}(t)$ is known as a radius vector or position vector.
Curves in space:

\[ \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t) \mathbf{\hat{i}} + y(t) \mathbf{\hat{j}} + z(t) \mathbf{\hat{k}} \]

is an example of a vector-valued function

**Definition:** A vector-valued function of \( t \) is a rule \( \mathbf{f} \) which assigns to each real number \( t \) (in some set of real numbers called the domain of \( \mathbf{f} \)) exactly one vector.
\[ \mathbf{f}(t) = f_1(t) \mathbf{i} + f_2(t) \mathbf{j} + f_3(t) \mathbf{k} \]

\(f_1(t), f_2(t), f_3(t)\) are the component functions of \(\mathbf{f}(t)\)

E.g. \(\mathbf{f}(t) = \cos t \mathbf{i} + \ln(4-t) \mathbf{j} + \sqrt{t+1} \mathbf{k}\)

**Example:** Find the domain of

\[ \mathbf{f}(t) = \cos t \mathbf{i} + \ln(4-t) \mathbf{j} + \sqrt{t+1} \mathbf{k} \]

**Domain of** \(\cos t\): \(-\infty < t < +\infty\)

**Domain of** \(\ln(4-t)\): \(4-t > 0 \Rightarrow 4 > t \Rightarrow t < 4\)

**Domain of** \(\sqrt{t+1}\): \(t+1 \geq 0 \Rightarrow t \geq -1\)

The domain of \(\mathbf{f}(t)\) is the intersection of the domains of the component functions.

**Solution:** \(t \geq -1\) and \(t < 4\) \(\Rightarrow -1 \leq t < 4\), \([-1, 4)\)
Graphs of vector-valued functions:

Examples:

1) \( \vec{r}(t) = 2t \hat{i} - 3 \hat{j} + (1 + 3t) \hat{k} \)

\[
\begin{align*}
  x(t) &= 2t \\
  y(t) &= -3 \\
  z(t) &= 1 + 3t
\end{align*}
\]

Line through \((0, -3, 1)\) parallel to \(\langle 2, 0, 3 \rangle\)

2) \( \vec{r}(t) = 3 \hat{i} + 2 \cos t \hat{j} + 2 \sin t \hat{k} \)

\[
\begin{align*}
  x(t) &= 3 \\
  y(t) &= 2 \cos t \\
  z(t) &= 2 \sin t
\end{align*}
\]

Recall: \(\cos^2 t + \sin^2 t = 1\) so \(\left(\frac{y}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = 1\)

\[
\frac{y^2}{4} + \frac{z^2}{4} = 1
\]

\(y^2 + z^2 = 4\) in \(3\)-space
Graph of \( \vec{r}(t) \) is the curve of intersection of 
the plane \( x=3 \) and the cylinder \( y^2+z^2=4 \), i.e. a circle.

\( \vec{r}(t) = \langle \cos t, \sin t, t \rangle \)

\[ \begin{align*}
x &= \cos t \\
y &= \sin t \\
z &= t
\end{align*} \]

\( x^2 + y^2 = 1 \) so \( \vec{r}(t) \) lies on cylinder \( x^2+y^2=1 \)

Graph of \( \vec{r}(t) \) is a helix