Chapter 3.8: Local Linear Approximations

Practice: Verify that the equation of the tangent line to \( f(x) = \sqrt[3]{x} \) at \( x_0 = 1 \)

\[ y = \frac{1}{3} x + \frac{2}{3} = L(x) \]

Note that if \( x_0 \) is close to 1, then \( f(x) \approx L(x) \)

\[ \sqrt[3]{x} \approx \frac{1}{3} x + \frac{2}{3} \quad \text{near} \ x_0 = 1 \]
We say that $L(x) = \frac{1}{3} x + \frac{2}{3}$ is the local linear approximation of $f(x) = \sqrt[3]{x}$ at $x_0 = 1$.

Note: $f(1.1) \approx L(1.1)$
To estimate \( \sqrt[3]{1.1} = f(1.1) \),

we use \( L(1.1) = \frac{1}{3} (1.1)^{2/3} \)

\[
= \frac{11}{30} + \frac{2}{3} = \frac{11}{30} + \frac{20}{30} = \frac{31}{30} = 1.033\bar{3}
\]

Calculator: \( \sqrt[3]{1.1} \approx 1.03228 \)

Note: If \( x_0 \) is not close to 1, the approximation is useless.

\( \sqrt[3]{8} = 2 \)

but \( \sqrt[3]{8} \approx \frac{1}{3} (8) + \frac{2}{3} = \frac{10}{3} = 3.33\bar{3} \) Terrible!
Example:

(a) Confirm that \( \frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x \) near \( x_0 = 0 \).

We need to show that the equation of the tangent line to \( f(x) = \frac{1}{\sqrt{1-x}} \) at \( x_0 = 0 \)

is \( y = 1 + \frac{1}{2}x \). (You do this as an exercise)
(b) Use this fact to approximate \((0.9)^{-\frac{1}{2}}\)

\[
(0.9)^{-\frac{1}{2}} = \frac{1}{\sqrt{0.9}} = \frac{1}{\sqrt{1-0.1}}
\]

Since \(\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x\)

then \(\frac{1}{\sqrt{1-0.1}} \approx 1 + \frac{1}{2}(0.1) = \frac{21}{20} = 1.05\)

Calculator: \((0.9)^{-\frac{1}{2}} \approx 1.0541\)
Example:

Approximate \( \sin 61^\circ \)

\[ f(x) = \sin x \]

\[ x_0 = 60^\circ = \frac{\pi}{3} \]

Now \( 61^\circ = 60^\circ + 1^\circ = \frac{\pi}{3} + \frac{\pi}{180} \)

\[ f'(x) = \cos x \]

\[ f'(\frac{\pi}{3}) = \cos \frac{\pi}{3} = \frac{1}{2} \]
\[ y - y_0 = m (x - x_0) \]
\[ m = \frac{1}{2} \]
\[ x_0 = \frac{\pi}{3} \]
\[ y_0 = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \]

\[ y - \frac{\sqrt{3}}{2} = \frac{1}{2} (x - \frac{\pi}{3}) \]

\[ y = \frac{1}{2} (x - \frac{\pi}{3}) + \frac{\sqrt{3}}{2} = l(x) \]

local linear approximation

of \( f(x) = \sin x \) at \( x_0 = \frac{\pi}{3} \)
So \( \sin 61^\circ = \sin \left( \frac{\pi}{3} + \frac{\pi}{160} \right) \)

\[ \approx \frac{1}{2} \left( \frac{\pi}{3} + \frac{\pi}{160} - \frac{\pi}{3} \right) + \frac{\sqrt{3}}{2} \]

\[ = \frac{\pi}{360} + \frac{\sqrt{3}}{2} \approx 0.87475 \]

Calculator: \( \sin 61^\circ \approx 0.87462 \)