Max/Min Problems

Goal: Find relative and absolute extrema of functions of two variables

Review:

- Open interval \((a, b)\)
  - Find relative extrema
    1. Find critical points (CP's) on \((a, b)\)
    2. Test CP's
       - E.g., 2nd Derivative Test

- Closed interval \([a, b]\)
  - Find absolute extrema
    1. Find CP's on \([a, b]\)
    2. Compare \(f\)-values of CP's from (1) to \(f\)-values of endpoints.
New stuff:

Open Region
no boundary points

Closed region
includes boundary points

Finding Relative Extrema on Open Regions:

Procedure:
1. Find critical points, i.e. points where

\[
\begin{align*}
f_x &= 0 \\
f_y &= 0
\end{align*}
\]
(2) Test CP's with Second Partial Test

\[ D(x, y) = f_{xx} f_{yy} - f_{xy} f_{yx} = f_{xx} f_{yy} - (f_{xy})^2 \]

Memory Trick: 2x2 determinant

\[ D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \]

Note: \( f_{xx} = \) concavity in \( x \)-direction
\( f_{yy} = \) concavity in \( y \)-direction

Theorem: If \( f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \), then

1. \( D(x_0, y_0) > 0 \) and \( f_{xx}(x_0, y_0) > 0 \) [or \( f_{yy}(x_0, y_0) > 0 \)]
   then relative minimum at \( (x_0, y_0) \)

2. \( D(x_0, y_0) > 0 \) and \( f_{xx}(x_0, y_0) < 0 \) [or \( f_{yy}(x_0, y_0) < 0 \)]
   then relative maximum at \( (x_0, y_0) \)

3. \( D(x_0, y_0) < 0 \) \( \Rightarrow \) saddle point at \( (x_0, y_0) \)

4. \( D(x_0, y_0) = 0 \) \( \Rightarrow \) test fails.
Examples: Find locations of all relative extrema and saddle points

1) \( f(x, y) = x^2 + y^2 \)

   1) Find CP's
   \[
   \begin{cases}
   f_x = 2x = 0 \\
   f_y = 2y = 0
   \end{cases} \Rightarrow \text{CP: } (0, 0)
   \]

   2) Test CP's
   \[
   f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 0
   \]
   \[D(x, y) = (2)(2) - 0^2 = 4\]
   \[D(0, 0) = 4 > 0 \quad \Rightarrow \text{relative min at } (0, 0)\]
   \[f_{xx}(0, 0) = 2 > 0\]
\( f(x, y) = 4xy - x^4 - y^4 \)

\[
\begin{cases}
  f_x = 4y - 4x^3 = 0 \\
  f_y = 4x - 4y^3 = 0
\end{cases}
\implies
\begin{align*}
  y - x^3 &= 0 \\
  x - y^3 &= 0
\end{align*}
\]

So \( y = x^3 \implies x - x^9 = 0 \) 
\( x(1 - x^8) = 0 \)

\( x = 0 \quad x = 1 \quad y = 0 \)
\( x = -1 \quad y = 1 \)

**CP's:** \( (0, 0) \), \( (1, 1) \), \( (-1, -1) \)

\[ f_{xx} = -12x^2 \quad f_{yy} = -12y^2 \quad f_{xy} = 4 \]

\[ D(x, y) = (-12x^2)(-12y^2) - 4^2 = 144x^2y^2 - 16 \]

\[ D(0, 0) = -16 < 0 \implies \text{saddle point at } (0, 0) \]

\[ D(1, 1) = 144 - 16 > 0 \text{ and } f_{xx}(1, 1) = -12 < 0 \]
\[ \implies \text{relative max at } (1, 1) \]

\[ D(-1, -1) = 144 - 16 > 0 \text{ and } f_{xx}(-1, -1) = -12 < 0 \]
\[ \implies \text{relative max at } (-1, -1) \]
Example: Find all points \((x,y,z)\) on the surface 
\[ z^2 - xy = 5 \] closest to origin

1. Write equation for quantity to be maximized or minimized
   Minimize distance 
   \[
   L = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}
   \]
   Easier to minimize 
   \[ L^2 = x^2 + y^2 + z^2 \]

2. Reduce to two variables
   \[
   z^2 - xy = 5 \implies z^2 = xy + 5
   \]
   So we want to minimize 
   \[ f(x,y) = x^2 + y^2 + xy + 5 \]

3. Proceed as in last two examples
   \[
   \begin{cases}
   f_x = 2x + y = 0 \implies y = -2x \\
   f_y = 2y + x = 0 \\
   \end{cases}
   \]
   So 
   \[
   -4x + x = 0 \implies x = 0, y = 0 \quad CP: (0,0) \]
\[ f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 1 \]

\[ \Delta(x, y) = (2)(2) - 1^2 = 3 \]

\[ \Delta(0, 0) = 3 > 0 \quad \text{and} \quad f_{xx}(0, 0) = 2 > 0 \]

So relative minimum at \((0, 0)\)

(4) Answer the question!

\[ x = 0, y = 0 \quad \Rightarrow \quad z^2 - xy = 5 \]

\[ z^2 - 0 = 5 \quad \Rightarrow \quad z = \pm \sqrt{5} \]

Solution: \((0, 0, \sqrt{5}), (0, 0, -\sqrt{5})\)
Finding Absolute Extrema on Closed Regions

Example: Find the absolute extrema of \( z = f(x, y) = x^2 + y^2 \) on region \( R : \frac{x^2}{4} + \frac{y^2}{9} \leq 1 \)

Illustration

Two possible locations for absolute extrema

(1) critical points inside region

(2) boundary points
Procedure:

1. Find CP's inside \( \frac{x^2}{y} + \frac{y^2}{9} < 1 \)

   \[ f(x,y) = x^2 + y^2 \]

   \[ f_x = 2x = 0 \qquad \rightarrow \quad \text{CP: } (0,0) \]

   \[ f_y = 2y = 0 \]

   which is inside \( \frac{x^2}{y} + \frac{y^2}{9} < 1 \)

   \[ f(0,0) = 0 \]

2. "Test" boundary points, i.e. parametrize boundary and convert to a Calc I problem.

Boundary:

\[ \frac{x^2}{4} + \frac{y^2}{9} = 1 \]

\[ \left( \frac{x}{2} \right)^2 + \left( \frac{y}{3} \right)^2 = 1 \]

\[ \frac{x}{2} = \cos t \quad \frac{y}{3} = \sin t \]

\[ x = 2 \cos t \quad y = 3 \sin t \quad 0 \leq t \leq 2 \pi \]
Evaluate $f(x,y)$ on the boundary

\[ z = f(2 \cos t, 3 \sin t) = (2 \cos t)^2 + (3 \sin t)^2 \]

\[ z = 4 \cos^2 t + 9 \sin^2 t \]

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Find absolute extrema of this on $[0, 2\pi]$.

\[ \frac{dz}{dt} = 8 \cos t \cdot (-\sin t) + 18 \sin t \cos t = 10 \sin t \cos t \]

(i) Find CP's on $0 < t < 2\pi$

(ii) Check endpoints $t = 0$, $t = 2\pi$

(i) $\frac{dz}{dt} = 0 \implies \sin t \cos t = 0$

\[ \sin t = 0 \quad \cos t = 0 \]

\[ t = \pi, \quad t = \frac{3\pi}{2} \]

\[ z(\pi) = 4(-1)^2 + 0 = 4 \]

\[ z(\frac{3\pi}{2}) = 4 \]
\[ z \left( \frac{\pi}{2} \right) = 0 + 9 \left( -1 \right)^2 \quad z \left( \frac{3\pi}{2} \right) = 9 \]

\[ z \left( \frac{3\pi}{2} \right) = 0 + 9 \left( -1 \right)^2 \quad z \left( \frac{3\pi}{2} \right) = 9 \]

\( \text{At endpoints:} \)

\[ z \left( 0 \right) = 4 \quad z \left( 2\pi \right) = 4 \]

**Compare all values:**

**Absolute Min of 0 at** \( t = 0, 0 \)

**Absolute Max of 9 at** \( t = \frac{\pi}{2}, \frac{3\pi}{2} \)

\[ x = 2 \cos t, \quad y = 3 \sin t \]

\( t = \frac{\pi}{2}: \quad (0, 3) \quad t = \frac{3\pi}{2}: \quad (0, -3) \)

**Exercise:** Read through Example 5 pages 982-983

\( R \) is a triangular region