Chapter 5.2 (Part One): Relative Extrema

Some terminology:
\( f'(x) \) changes from increasing to decreasing at a relative (local) maximum
\( f'(x) \) changes from decreasing to increasing at a relative (local) minimum

A relative extremum is either a relative maximum or a relative minimum.

The plural of extremum is extrema.
Goal: Given \( f(x) \), find all relative extrema.

Strategy:

- Find the domain of \( f \)
- Find \( f'(x) \)
- Find all values of \( x \) (on the domain of \( f \)) at which \( f'(x) = 0 \) or \( f'(x) \) ONE. These values are called the critical points of \( f \) and are the only values where \( f'(x) \) might change sign.
- Test the critical points with either the First Derivative Test or the Second Derivative Test.
**First Derivative Test (FDT)**

This is what we did in Chapter 5.1.

**Recall:** Example (1) from last Thursday

\[ f(x) = x^4 - 2x^3 \]

\[ f'(x) = 0 \text{ at } x = 0, x = \frac{3}{2} \]

By FDT:

\[ f'(x) \]

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>(\frac{3}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decrease</td>
<td></td>
<td>Increase</td>
</tr>
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\[ f(x) \text{ Decrease, Decrease, Increase} \]
Relative min at $x = \frac{3}{2}$

No relative extremum at $x = 0$

**Note:** The actual extrema are $y$-coordinates.

So we compute $f\left(\frac{3}{2}\right) = \frac{-27}{16}$

So $f$ has a relative min of $\frac{-27}{16}$ at $x = \frac{3}{2}$

The locations of the extrema are the $x$-coordinates.

Be aware of what the question is asking.
Recall: Example (2) from last Thursday

\[ f(x) = x^\frac{1}{3} (x+3)^\frac{2}{3} \]

\[ f'(x) \text{ ONE at } \begin{cases} x = 0, \ x = -3 \end{cases} \]

\[ f'(x) = 0 \text{ at } x = -1 \]

By FDT:

\[ f'(x) \begin{array}{c|c|c|c|c} -3 & -1 & 0 & \\
\hline\end{array} \begin{array}{c} + \ \text{ONE} \ - \ 0 \ + \ \text{ONE} \ + \end{array} \]

\[ f(x) \begin{array}{c|c|c|c|c} -3 & -1 & 0 & \\
\hline\end{array} \begin{array}{c} \text{Incr} \ \text{Decr} \ \text{Incr} \ \text{Incr} \end{array} \]

Relative max at \( x = -3 \); Relative min at \( x = -1 \)
New examples

(1) \( f(x) = \frac{\ln x}{x} \)

Find locations of all relative extrema

Domain: \((0, +\infty)\)

Find critical points

\[
f'(x) = x \left( \frac{1}{x} \right) - \ln x = \frac{1 - \ln x}{x^2}
\]

\( f'(x) \) exists everywhere on domain of \( f \)
\[ f'(x) = 0 \iff \frac{1 - \ln x}{x^2} = 0 \]
\[ \iff 1 - \ln x = 0 \]
\[ \iff 1 = \ln x \]
\[ \iff \boxed{x = e} \text{ critical point} \]

By FDT:

\[ \begin{array}{ccc}
0 & \rightarrow & e \\
\downarrow & + & \downarrow \\
\frac{d}{dx}f(x) & \text{Incr} & f(x) \text{ Decr}
\end{array} \]
\[ f'(x) = \frac{1 - \ln x}{x^2} \] always positive

\[ x = 1: \quad f'(1) = \frac{1 - \ln 1}{(pos)} = \frac{1 - 0}{(pos)} > 0 \]

\[ x = e^2: \quad f'(e^2) = \frac{1 - \ln e^2}{(pos)} = \frac{1 - 2}{(pos)} < 0 \]

Relative max at \( x = e \)
Second Derivative Test (SOT)

Cons:
- Need to compute $f''(x)$
- Only works for $x = x_0$ if $f'(x_0) = 0$
  (These values are called stationary points)
- Doesn't work for $x = x_0$ if $f'(x_0) \neq 0$
- Even if $f'(x_0) = 0$, test may be inconclusive

Pros:
- Easier to apply than FOT.
How it works

If \( f'(x_0) = 0 \) (i.e., \( x = x_0 \) is a stationary point), then evaluate \( f''(x_0) \). Three cases:

**Case 1:** \( f''(x_0) > 0 \) (so \( f(x) \) is concave up on some interval including \( x_0 \))

Relative min at \( x = x_0 \)
Case 2: \( f''(x_0) < 0 \)

Relative max at \( x = x_0 \)

Case 3: \( f''(x_0) = 0 \)

Inconclusive. Must use FDT.
Example (1) Revisited

(1) \[ f(x) = \frac{\ln x}{x} \]

\[ f'(x) = \frac{1 - \ln x}{x^2} \]

\[ f''(x) = 0 \quad \text{at} \quad x = e \]

Stationary point

\[ f''(x) = \frac{2 \ln x - 3}{x^3} \]

Exercise: Verify
By Second Derivative Test:

$$f''(e) = \frac{2\ln e - 3}{e^3} = \frac{2 - 3}{e^3} = \frac{\text{neg}}{\text{pos}} < 0$$

Therefore, there is a relative max at $x = e$. 
\( f(x) = \frac{1}{2}x - \sin x \)

Find locations of all relative extrema on \([0, 2\pi]\)

\[ f'(x) = \frac{1}{2} - \cos x \]

\( f'(x) \) exists everywhere

\[ f'(x) = 0 \iff \frac{1}{2} - \cos x = 0 \]

\[ \iff \frac{1}{2} = \cos x \]

\[ x = \frac{\pi}{3}, \quad x = \frac{5\pi}{3} \]

Stationary points
\[ f''(x) = \sin x \]

By SOT:
\[ f''\left(\frac{\pi}{3}\right) = \sin \frac{\pi}{3} > 0 \]

\[ x = \frac{\pi}{3} \]

Relative min at \( x = \frac{\pi}{3} \)

\[ f''\left(\frac{5\pi}{3}\right) = \sin \frac{5\pi}{3} < 0 \]

Relative max at \( x = \frac{5\pi}{3} \)
Chapter 5.4: Absolute Extrema

Definitions:

$f(x)$ has an **absolute maximum** at $x = x_0$ if $f(x_0) \geq f(x)$ for all $x$ in the domain of $f$

$f(x)$ has an **absolute minimum** at $x = x_0$ if $f(x_0) \leq f(x)$ for all $x$ in the domain of $f$

Note: The absolute extrema are $y$-values, the locations of the absolute extrema are $x$-values.
Illustration: Absolute vs Relative Extrema
Not all functions have absolute extrema, e.g.

No absolute max

No absolute min

Neither

However, there is a special case that guarantees the existence of both an absolute max and an absolute min.
Extreme Value Theorem (EVT):

If \( f \) is continuous on a closed interval \([a,b]\), then \( f \) has both an absolute max and an absolute min.

These absolute extrema must occur at one of the following:

1. a critical point in the open interval \((a,b)\)
2. an endpoint \( x=a \) or \( x=b \).

Illustration:

Think: Why is continuity on \([a,b]\) necessary?
Goal: Given continuous function $f$ on $[a,b]$, find the absolute extrema of $f$ on $[a,b]$

Strategy:

- Find critical points in open interval $(a,b)$
- Compare $y$-values of any such critical points and the endpoints $x=a$ and $x=b$
- Absolute max = largest $y$-value
  Absolute min = smallest $y$-value
Examples

(1) Find all absolute extrema, and where they occur, for \( f(x) = x^3 - 3x - 1 \) on \([0, 2]\)

\( f \) is continuous on \([0, 2]\) \implies \text{use EVT}

Find critical points in \((0, 2)\):

\[
f'(x) = 3x^2 - 3 = 3(x-1)(x+1)
\]

is defined everywhere

\( f'(x) = 0 \) at \( x = 1, x = -1 \)

\( x = -1 \) \( \text{not in } (0, 2) \)
Compare $y$-values:

\[ f(1) = 1 - 3 - 1 = -3 \]
\[ f(0) = -1 \]
\[ f(2) = 8 - 6 - 1 = 1 \]

$f$ has an absolute max of $1$ at $x = 2$
$f$ has an absolute min of $-3$ at $x = 1$
(2) Find all absolute extrema, and where they occur, for \( f(x) = \sin x - \cos x \) on \([0, \pi]\)

\( f \) is continuous on \([0, \pi]\) \( \Rightarrow \) use EVT

Find critical points in \((0, \pi)\):

\( f'(x) = \cos x + \sin x \) is defined everywhere

\( f'(x) = 0 \) \( \Rightarrow \) \( \cos x = -\sin x \) \( \Rightarrow \) \( x = \frac{3\pi}{4} \)
Compare $y$-values:

$f(0) = \sin 0 - \cos 0 = -1$

$f(\pi) = \sin \pi - \cos \pi = 1$

$f(\frac{3\pi}{4}) = \sin \frac{3\pi}{4} - \cos \frac{3\pi}{4}
= \frac{\sqrt{2}}{2} - (-\frac{\sqrt{2}}{2}) = \frac{2\sqrt{2}}{2} = \sqrt{2}$

Absolute max: \( \left( \frac{3\pi}{4}, \sqrt{2} \right) \)

Absolute min: \( (0, -1) \)