Chapter 14.4: Surface Area and Parametric Surfaces

Goal: Find the surface area $S$ of a surface $z = f(x, y)$ above a closed region $R$ in the $xy$-plane.

Strategy:

Divide $R$ into $n$ rectangular boxes. Arbitrary rectangle $R_k$ has area $\Delta A_{R_k} = \Delta x_k \Delta y_k$
Approximate area of the patch $\sigma_k$ of the surface above $R_k$.

Area of $\sigma_k$ = area of parallelogram $\tau_k$ (piece of tangent plane). $\tau_k$ is determined by two vectors: $\vec{q}_{ik}$ and $\vec{r}_k$.

We need the components of $\vec{q}_{ik}$ and $\vec{r}_k$. 
\( \vec{r}_k = \langle 0, \Delta y_k, ? \rangle \)

To get the \( z \)-component of \( \vec{r}_k \), translate \( \vec{r}_k \) so its tail is at the origin.

\[ \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\partial z}{\partial y} = \frac{?}{\Delta y_k} \]

So \( ? = \frac{\partial z}{\partial y} \Delta y_k \)

So \( \vec{r}_k = \langle 0, \Delta y_k, \frac{\partial z}{\partial y} \Delta y_k \rangle \)

Similarly, \( \vec{q}_k = \langle \Delta x_k, 0, \frac{\partial z}{\partial x} \Delta x_k \rangle \)
Now
\[ \vec{q}_{lc} \times \vec{r}_{lc} = \left| \begin{array}{ccc} \vec{z} & \vec{j} & \vec{k} \\ \Delta x_k & \frac{\partial z}{\partial x} \Delta x_k & 0 \\ 0 & \Delta y_k & \frac{\partial z}{\partial y} \Delta y_k \end{array} \right| = \left( -\frac{\partial z}{\partial x} \vec{z} - \frac{\partial z}{\partial y} \vec{j} + \vec{k} \right) \frac{\Delta x_k \Delta y_k}{\Delta A_k} \]

So area of \( \mathcal{C}_k = \| \vec{q}_{lc} \times \vec{r}_{lc} \| = \sqrt{\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 + 1} \Delta A_k \)

We now add up all of these areas to get an approximation for \( S \)
\[ S = \sum_{k=1}^{n} \sqrt{\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 + 1} \Delta A_k \]

To get an exact value of \( S \), let \( n \to \infty \)
\[ S = \lim_{n \to +\infty} \sum_{k=1}^{n} \sqrt{\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 + 1} \Delta A_k \]
So \[ S = \iint_R \sqrt{\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 + 1} \, dA \]

Example: Find the surface area of the portion of \( z = x^2 + y^2 \) below \( z = 1 \).

\[ \frac{\partial z}{\partial x} = 2x \quad \frac{\partial z}{\partial y} = 2y \]
\[ S = \iint_R \sqrt{4x^2 + 4y^2 + 1} \, dA \]

\[ x^2 + y^2 = 1 \quad (r = 1) \]

Use polar coordinates

\[ S = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1} \, r \, dr \, d\theta \]

\[ = \cdots = \frac{\pi}{6} (5\sqrt{5} - 1) \quad [\text{Verify}] \]

\[ u = 4r^2 + 1 \]
Parametric Surfaces

Recall: The graph of $\mathbf{r}(t) = (x(t), y(t), z(t))$ is a curve in space, i.e., the graph with parametric equations:

\[
\begin{align*}
    x &= x(t) \\
    y &= y(t) \\
    z &= z(t)
\end{align*}
\]

E.g., $\begin{cases} x = \cos t \\ y = \sin t \\ z = 6 \end{cases}$ is a helix.

New stuff: The graph of a vector-valued function of two variables $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ is a surface in space, i.e., the surface with parametric equations:

\[
\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}
\]
Example: Identify the parametric surface

\[ X = u \cos \nu, \quad Y = u, \quad Z = u \sin \nu \]

Eliminate the parameters:

\[ X = y \cos \nu, \quad Z = y \sin \nu \]

\[ x^2 + z^2 = (y \cos \nu)^2 + (y \sin \nu)^2 = y^2 \cos^2 \nu + y^2 \sin^2 \nu = y^2 \]

Core about y-axis

Example: Give parametric equations for the surface \( x^2 + y^2 + z^2 = 25 \)

Sphere — can be written as \( \rho = 5 \)

Now

\[ \begin{align*}
  x &= \rho \sin \phi \cos \theta \\
  y &= \rho \sin \phi \sin \theta \\
  z &= \rho \cos \phi
\end{align*} \]

\[ \rho = 5 \] \quad \Rightarrow \quad \begin{align*}
  X &= 5 \sin \phi \cos \theta \\
  Y &= 5 \sin \phi \sin \theta \\
  Z &= 5 \cos \phi
\end{align*} \]

Useful for graphing utilities: see pg 1029 - 1030
We can describe other types of "unusual" surfaces with parametric equations, e.g.

\[
\begin{align*}
\begin{cases}
  x &= (a+b \cos v) \cos u \\
  y &= (a+b \cos v) \sin u \\
  z &= b \sin v
\end{cases}
\end{align*}
\]
Torus  pg 1038

\(a, b\) constants  \(0 \leq u \leq 2\pi, \ 0 \leq v \leq 2\pi\)

Finding Tangent Planes to Parametric Surfaces

Recall:

\(P: (x_0, y_0, z_0)\) point where \(t = t_0\)

\(-\text{graph of } \vec{r}(t)\)

New stuff:  Definition: Given \(\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))\)

\[
\begin{align*}
\frac{\partial \vec{r}}{\partial u} &= \vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \\
\frac{\partial \vec{r}}{\partial v} &= \vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle
\end{align*}
\]
The vector normal to the tangent plane is
\[
\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}
\]
Example: Find the tangent plane to 

\[ x = u^2, \ y = v^2, \ z = u + v \] 

at the point \( (1, 4, 3) \)

\[ \frac{\partial x}{\partial u} = 2u, \quad \frac{\partial y}{\partial u} = 0, \quad \frac{\partial z}{\partial u} = 1 \]

So \( \mathbf{r}_u = \langle 2u, 0, 1 \rangle \)

\[ \frac{\partial x}{\partial v} = 0, \quad \frac{\partial y}{\partial v} = 2v, \quad \frac{\partial z}{\partial v} = 1 \]

So \( \mathbf{r}_v = \langle 0, 2v, 1 \rangle \)

Need \( (u, v) \) that corresponds to \( (1, 4, 3) \)

\[ u^2 = 1, \ v^2 = 4, \ u + v = 3 \quad \Rightarrow \quad (u, v) = (1, 2) \]

So \( \mathbf{r}_u (1, 2) = \langle 2, 0, 1 \rangle \) and \( \mathbf{r}_v (1, 2) = \langle 0, 4, 1 \rangle \)

Now \( \langle 2, 0, 1 \rangle \times \langle 0, 4, 1 \rangle = \langle -4, -2, 8 \rangle \) [Verify]

So tangent plane: \( -4(x-1) - 2(y-4) + 8(z-3) = 0 \)

\[ \therefore -2x - y + 4z = 6 \]
Exercise: Find the tangent plane to 
\[ \mathbf{r}(u,v) = u \mathbf{e}_i + u e^v \mathbf{e}_j + v e^u \mathbf{e}_k \]
at the point where \( u = \ln 2, v = 0 \)

Solution: \( 2x - (\ln 2)z = 0 \)
Finding Surface Areas of Parametric Surfaces

Goal: Find the surface area $S$ of a parametric surface $\vec{r}(u,v)$ above a closed region $R$ in the $uv$-plane.

Strategy:

Divide $R$ into $n$ rectangular boxes. Arbitrary rectangle $R_k$ has area $\Delta A_k = \Delta u_k \Delta v_k$

Approximate area of the patch $\sigma_k$ of surface above $R_k$.

Area of $\sigma_k \approx$ area of parallelogram $\Sigma_k$ (piece of tangent plane) determined by $\frac{\partial \vec{r}}{\partial u} \Delta u_k$ and $\frac{\partial \vec{r}}{\partial v} \Delta v_k$
Area of $\mathcal{C}_k = \left\| \frac{\partial \mathbf{r}}{\partial u} \Delta u_k \times \frac{\partial \mathbf{r}}{\partial v} \Delta v_k \right\|

= \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \frac{\Delta u_k \Delta v_k}{\Delta A_k}

So surface area

$$S = \lim_{n \to +\infty} \sum_{k=1}^{n} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta A_k$$

So

$$S = \iint_{R} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, dA$$

Note: $dA$ will become $dudv$ or $dvdu$ when converting to an iterated integral.
Example. Find surface area of

\[ x = u \cos v, \quad y = u, \quad z = u \sin v; \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi \]

\[
\frac{\partial x}{\partial u} = \cos v, \quad \frac{\partial y}{\partial u} = 1, \quad \frac{\partial z}{\partial u} = \sin v
\]

\[
\frac{\partial x}{\partial v} = -u \sin v, \quad \frac{\partial y}{\partial v} = 0, \quad \frac{\partial z}{\partial v} = u \cos v
\]

\[
\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\cos v & \sin v & 0 \\
-u \sin v & 0 & u \cos v
\end{vmatrix} = \vec{r} = \langle u \cos v, -u, u \sin v \rangle
\]

\[
\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| = \sqrt{\left(u \cos v\right)^2 + (-u)^2 + (u \sin v)^2} = \cdots = \sqrt{2} \cdot u
\]

\[
S = \iiint_0^{2\pi} \iiint_0^2 \sqrt{2} \cdot u \, du \, dv = \cdots = 4\pi \sqrt{2}
\]

Note: \[ S = \iiint_0^{2\pi} \iiint_0^2 \sqrt{2} \cdot u \, dv \, du \] is OK too.