7) Prove ∀ n > 0, \( 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \) 

**Base Case:** Prove \( P(1) \)

\[
1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{(2)(3)}{6} = 1 \quad \checkmark
\]

\( \therefore P(1) \) is true

**Inductive Step:**

Assume \( P(k) \): \( 1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \) for some \( k > 0 \)

Show \( P(k+1) \): \( 1^2 + 2^2 + \ldots + (k+1)^2 \)

\[
= \frac{(k+1)(k+2)(2k+3)}{6}
\]

So \( 1^2 + 2^2 + \ldots + (k+1)^2 = 1^2 + 2^2 + \ldots + k^2 + (k+1)^2 \)

\[
= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{by inductive hypothesis}
\]

\[
= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6}
\]
\[
\frac{(k+1) \left[ 2k^2 + 7k + 6 \right]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}
\]

QED

29) Prove that \( \frac{n^2 > 5n+10}{P(n)} \) for \( n \geq 6 \)

**Base Case:** Prove \( P(7) \).
\[
7^2 > 5 \cdot 7 + 10
\]
\[
49 > 45 \quad \checkmark \quad \therefore \ P(7) \text{ is true}
\]

**Inductive Step:**

Assume \( P(k) \): \( k^2 > 5k + 10 \) for some \( k \geq 6 \)

Show \( P(k+1) \): \( (k+1)^2 > 5(k+1) + 10 \)

So \( (k+1)^2 = k^2 + 2k + 1 \)

\[
> (5k + 10) \quad + \quad 2k + 1 \quad \text{(by inductive hypothesis)}
\]
\[
> 5k + 10 + 12 + 1 \quad \text{(because } k \geq 6\text{)}
\]
\[
> 5k + 15
\]
\[
= 5(k+1) + 10
\]

QED
43) Prove that \( 7^n - 2^n \) is divisible by 5 for \( n > 0 \)

\[ P(n) \]

**Base case:** Prove \( P(1) \). \( 7^1 - 2^1 = 5 \), which is clearly divisible by 5. \( \therefore P(1) \) is true.

**Inductive step:**
Assume \( P(k) \): \( 7^k - 2^k \) is divisible by 5 for some \( k > 0 \)

\[ \Rightarrow 7^k - 2^k = 5m \text{ for some } m \in \mathbb{Z} \]

\[ \Rightarrow 7^k = 5m + 2^k \]

**Show \( P(k+1) \):** \( 7^{k+1} - 2^{k+1} \) is divisible by 5.

So \( 7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k \)

\[ = 7(5m + 2^k) - 2 \cdot 2^k \text{ (by inductive hypothesis)} \]

\[ = 35m + 7 \cdot 2^k - 2 \cdot 2^k \]

\[ = 35m + 2^k (7 - 2) \]

\[ = 35m + 2^k (5) \]

\[ = 5(7m + 2^k) \]

Since \( (7m + 2^k) \in \mathbb{Z} \),

\( 7^{k+1} - 2^{k+1} \) is divisible by 5.

QED
67) Prove that any amount of postage \( \geq 12 \) cents can be built using only 4-cent and 5-cent stamps.

**Informal argument:** Let \( P(n) \) be the statement that only 4-cent and 5-cent stamps are needed to build \( n \) cents worth of postage. If \( P(k-3) \) is true, then \( P(k+1) \) is true since if \((k-3)\) cents can be paid in this way, then \((k+1)\) cents can be paid in this way as well by simply adding an additional 4-cent stamp.

So we need the fourth previous statement, \( P(k-3) \), to prove \( P(k+1) \). So we will need 4 base cases. See pg 104 for a similar example.

**Formal proof:** Let \( P(n) \) be the statement that only 4-cent and 5-cent stamps are needed to build \( n \) cents worth of postage.

**Base Cases** Prove \( P(12), P(13), P(14), P(15) \)

\[
12 = 4 + 4 + 4; \quad 13 = 5 + 4 + 4; \quad 14 = 5 + 5 + 4; \quad 15 = 5 + 5 + 5
\]

\[
\therefore P(12), P(13), P(14), P(15) \text{ are all true.}
\]

**Inductive Step:** Assume \( P(r) \) true for all \( r, 12 \leq r \leq k \). Prove \( P(k+1) \).

Assume \( k+1 \geq 16 \) since we proved \( P(r) \) for \( r = 12, 13, 14, \) and 15. If \( k+1 \geq 16 \) then \( k-3 \geq 12 \) so by the inductive hypothesis \( P(k-3) \) is true, which means \((k-3)\) cents of postage is attainable with 4-cent and 5-cent stamps. Adding one more 4-cent stamp shows \((k+1)\) cents of postage is attainable in this way as well.

So \( P(k+1) \) is true.

QED