54) Let $\mathcal{B}$ be collection of sets. Define binary relation $\rho$ on $\mathcal{B}$: For $S, T \in \mathcal{B}$, $S \rho T$ is equivalent to $T$.
Prove $\rho$ is an equivalence relation on $\mathcal{B}$.
Must show $\rho$ is reflexive, symmetric, and transitive.

**Reflexive:** Must show $\forall S \in \mathcal{B}, S \rho S$

$\Rightarrow S$ is equivalent to $S \Rightarrow \exists$ bijection $f: S \to S$

We can use the identity permutation $f: S \to S$
defined by $f(s) = s$, $\forall s \in S$ (i.e. $f$ maps every element of $S$ to itself). So we have found the required bijection.

**Symmetric:** Must show $\forall S, T \in \mathcal{B}, S \rho T \Rightarrow T \rho S$

Assume $S \rho T \Rightarrow S$ is equivalent to $T$

$\Rightarrow \exists$ bijection $f: S \to T$

$\Rightarrow \exists$ bijection $f^{-1}: T \to S$ (all bijections $f$ have an inverse)

$\Rightarrow T$ is equivalent to $S$

$\Rightarrow T \rho S$
Transitive: Must show \( \forall S, T, U \in \mathcal{B} \)
\[(S \cap T \text{ and } T \cap U) \Rightarrow S \cap U\]

Assume \( S \cap T \text{ and } T \cap U \)

\( \Rightarrow S \) is equivalent to \( T \) and \( T \) is equivalent to \( U \)

\( \Rightarrow \exists \text{ bijection } f : S \rightarrow T \text{ and } \exists \text{ bijection } g : T \rightarrow U \)

\( \Rightarrow \exists \text{ bijection } g \circ f : S \rightarrow U \) (the composition of two bijections is a bijection)

\( \Rightarrow S \) is equivalent to \( U \)

\( \Rightarrow S \cap U \)

\( \therefore \rho \) is an equivalence relation on \( \mathcal{B} \).

55) Group the sets:

\( A = \{2, 4, 6\}, \ B = \mathbb{N}, \ C = \{x \mid x \in \mathbb{N} \text{ and } \exists y \in \mathbb{N} \text{ and } x = 2 \times y\}, \)

\( D = \{a, b, c, d, e\}, \ E = \mathcal{P}(\{1, 2, 3\}) \), \( F = \mathbb{Q}^+ \)

into equivalence classes according to \( \rho \) from previous problem.

Sets \( A, D, E \) are finite and thus will be equivalent if they have the same cardinality.

Now \(|A| = 3, |D| = 5, |E| = 2^3 = 8\) so \( D \) and \( E \) belong to the same equivalence class.

Infinite sets \( B \) and \( F \) are equivalent because set \( F = \mathbb{Q}^+ \) is countable (we have proven this earlier); hence \( B \) and \( F \) are in the same equivalence class.
Set \( C \) is the set of non-negative even integers. We can create a bijection \( f : B \to C \) defined as \( f(x) = 2x \). Thus \( B \) and \( C \) are equivalent and are in the same equivalence class.

To sum up:

\[ [A] = [A_3], \quad [B] = [C_1] = [E_1] = [B, C_3, F] \]
\[ [D] = [E_2] = [D, E_3] \]

61) Prove \( f = O(g) \) if \( f(x) = 3x^3 - 7x \), \( g(x) = \frac{x^3}{2} \)

Most find positive constants \( C_1, C_2, n_0 \) such that

\[ \forall x \geq n_0, \ C_1 g(x) \leq f(x) \leq C_2 g(x) \]

Let \( C_1 = 1 \), \( C_2 = 8 \)

So we have \( \frac{1}{2} x^3 \leq 3x^3 - 7x \leq 4x^3 \)

This is true for all \( x \geq 2 \). So let \( n_0 = 2 \).

66) Use limit test to prove \( f = O(g) \)

\[ \lim_{x \to \infty} \frac{3x^3 - 7x}{\frac{1}{2} x^3} = \lim_{x \to \infty} \frac{9x^2 - 7}{\frac{3}{2} x^2} \]

\[ = \lim_{x \to \infty} \frac{18x}{3x} = \frac{18}{3} \]

Since \( \lim_{x \to \infty} \frac{f(x)}{g(x)} \) is a positive real number, \( f = O(g) \)
Section 5.1

a) Is the graph simple?
   Yes, it has no loops and no parallel arcs.

b) Is the graph complete?
   No, every node is not adjacent to every other node.
   Ex: 1 is not adjacent to 3

c) Is the graph connected?
   Yes, there is a path from every node to every other node.

d) Can you find two paths from 3 to 6?
   Yes. 3, a_5, 5, a_6, 6 and 3, a_3, 4, a_4, 5, a_6, 6

e) Can you find a cycle?
   Yes. 3, a_3, 4, a_4, 5, a_5, 3

f) Can you find an arc whose removal will make the graph acyclic?
   Yes. Remove a_5. Now there is no path from any node back to itself without repeating an arc.

g) Can you find an arc whose removal will make the graph not connected?
   Yes. Remove arc a_1. Now there is no path to or from node 1.
5(a) Draw $K_6$

(b) Draw $K_{3,4}$