WA 7: Solutions

4. (a) Let \( f \) be a non-constant analytic function on the closed region \( R \), and \( f(z) \neq 0 \) for all \( z \in R \). Show that \( \min_{z \in R} |f(z)| \) can be attained only at a boundary point of \( R \).

(b) Let \( f \) be a non-constant analytic function on and inside a simple closed contour \( C \), and \( |f(z)| \) has a constant value on \( C \). Show that \( f \) has at least one zero inside \( C \).

Solution. (a) Since \( f(z) \neq 0 \) for all \( z \in R \), \( g(z) = 1/f(z) \) is analytic in \( R \). By the maximum principle, \( \max_{z \in R} |g(z)| \) can be attained only at a boundary point of \( R \). Then \( \min_{z \in R} |f(z)| = 1/\max_{z \in R} |g(z)| \) can be attained only at a boundary point of \( R \).

(b) Suppose \( f(z) \neq 0 \) for all \( z \) inside \( C \). Then by part (a) \( |f(z)| \) attains its minimum on \( C \). On the other hand, by the maximum principle \( |f(z)| \) attains its maximum on \( C \). Since \( |f(z)| \) is constant on \( C \), minimum and maximum values on and inside \( C \) are equal. Therefore, \( |f(z)| \) is a constant function which is possible (again, by the maximum principle or directly from the Cauchy–Riemann equations) only if \( f(z) \) is constant. The latter contradicts the assumption. Therefore, it is impossible that \( f(z) \neq 0 \) for all \( z \) inside \( C \).

5. Let \( f \) be an entire function such that \((1 + |z|^k)^{-1} f^{(m)}(z)\) is bounded for some positive integers \( k \) and \( m \). Prove that \( f^{(n)}(z) \) is identically zero for sufficiently large \( n \). How large must \( n \) be in terms of \( k \) and \( m \)?

Solution. Since \( f \) is entire, so is \( f^{(m)} \). By the Cauchy inequalities for \( g = f^{(m)} \), one has

\[
|g^{(j)}(0)| \leq \frac{j!M_R}{R^j}
\]

for every \( R > 0 \) and \( z_0 \in \mathbb{C} \), where \( M_R = \max_{|z - z_0| = R} |g(z)| \). Since

\[
(1 + |z|^k)^{-1} |f^{(m)}(z)| \leq K
\]

for some constant \( K \) and all \( z \in \mathbb{C} \), we have for \( j > k \) that

\[
|f^{(m+j)}(z_0)| = |g^{(j)}(z_0)| \leq \frac{j!(1 + |z_0| + R)^k K}{R^j} \rightarrow 0 \quad \text{as } R \rightarrow \infty.
\]

Thus \( f^{(m+j)}(z_0) = 0 \). Since \( z_0 \) was arbitrary, we conclude that \( f^{(n)}(z) \equiv 0 \) for \( n > m + k \).

6. Suppose that \( f \) is analytic on and inside the unit circle \( \gamma = \{ z \in \mathbb{C} : |z| = 1 \} \). Let \( \ell \) be the length of the image of \( \gamma \) under \( f \). Show that \( \ell \geq 2\pi|f'(0)| \).

Solution. Since \( f \) is analytic on and inside the unit circle \( \gamma \), so is \( f' \). Therefore, by the Cauchy integral formula applied to \( f' \) we have

\[
f'(0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(s)}{s} \, ds = \frac{1}{2\pi} \int_0^{2\pi} f'(e^{i\theta}) \, d\theta.
\]

Therefore,

\[
|f'(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f'(e^{i\theta})| \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} |\{f(e^{i\theta})\}'| \, d\theta = \frac{\ell}{2\pi}.
\]