Math 200
Exam 1
May 1st, 2019

Name: Solutions
Section:

* For free response questions, you must show all work. Answers without proper justification will not receive full credit. Partial credit will be awarded for significant progress towards the correct answer. Cross off any work that you do not want graded.

* This is a closed-book exam. You may not use any books or notes on this exam.

* You have 50 minutes to complete this exam. When time is called, stop writing immediately and turn in your exam at the front of the room - be sure to place your exam in the appropriate folder.

* You may not use any electronic devices including (but not limited to) calculators, cell phones, or tablets. Using such a device will be considered a violation of the university’s academic integrity policy and, at the very least, will result in a grade of 0 for this exam.

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1. (15 points) Find an equation for the plane containing the point \( A(1, 2, 3) \) and the line \( L \), given below.

\[
L : \begin{cases}
  x = 3 - t \\
  y = 2 + 2t \\
  z = t
\end{cases}
\]

Plugging \( t = 0 \) into \( L \) we get the point \( B(3, 2, 0) \).

The vector \( \overrightarrow{AB} = \langle 2, 0, -3 \rangle \) is also on the plane.

\[
\overrightarrow{n} = \langle 2, 0, -3 \rangle \times \langle -1, 2, 1 \rangle
= \begin{vmatrix}
  \hat{i} & \hat{j} & \hat{k} \\
  2 & 0 & -3 \\
  -1 & 2 & 1
\end{vmatrix}
= \langle 6, 1, 4 \rangle
\]

Plane: \( 6(x - 1) + (y - 2) + 4(z - 3) = 0 \)
2. (15 points) Compute all of the second-order partial derivatives of $f$.

$$f(x, y) = 3x^2y^4 - xe^y$$

$$\begin{align*}
\frac{\partial}{\partial x} f(x, y) &= 6x y^4 - e^y \\
\frac{\partial}{\partial y} f(x, y) &= 12x^2 y^3 - xe^y \\
\frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} f(x, y) \right) &= 6y^4 \\
\frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} f(x, y) \right) &= 24xy^3 - e^y \\
\frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} f(x, y) \right) &= 24xy^3 - e^y \\
\frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} f(x, y) \right) &= 36x^2y^2 - xe^y
\end{align*}$$
3. Consider the vector-valued function \( \mathbf{r}(t) = (3t, \ln t, t^2) \).

(a) (8 points) Find a set of parametric equations for the line tangent to \( \mathbf{r} \) at the point \( A(3,0,1) \). We'll call this line \( L \).

(b) (7 points) Compute the angle between the line \( L \) and the plane \( P : x + z = 3 \). You may leave your answer in terms of an inverse trig function.

\[
\begin{align*}
3t &= 3 \quad \text{\( t = 1 \)} \\
\ln t &= 0 \quad \text{\( t = 1 \)} \\
t^2 &= 1 \quad \text{\( t = \pm 1 \)}
\end{align*}
\]

Only \( t=1 \) works in all 3 equations.

\[
\mathbf{r}'(t) = \langle 3, \frac{1}{t}, 2t \rangle \\
\mathbf{r}'(1) = \langle 3, 1, 2 \rangle
\]

The normal vector to \( P \) is \( \mathbf{n} = \langle 1, 0, 1 \rangle \).

We can use the dot product to compute the angle between \( \mathbf{n} \) and \( \mathbf{r}'(1) \):

\[
\cos \Theta = \frac{\mathbf{n} \cdot \mathbf{r}'(1)}{\| \mathbf{n} \| \| \mathbf{r}'(1) \|}
\]

\[
= \frac{\langle 1, 0, 1 \rangle \cdot \langle 3, 1, 2 \rangle}{\| \langle 1, 0, 1 \rangle \| \| \langle 3, 1, 2 \rangle \|}
\]

\[
= \frac{5}{\sqrt{2} \sqrt{14}}
\]

\[
\Theta = \frac{\pi}{2} - \cos^{-1} \left( \frac{5}{\sqrt{2} \sqrt{14}} \right)
\]
4. (15 points) Determine whether the following lines are parallel, intersecting, or skew. If the lines intersect, determine the point of intersection.

\[ L_1: \begin{cases} x = 2 - t \\ y = 3t \\ z = 1 + 2t \end{cases} \quad L_2: \begin{cases} x = 4 - t \\ y = 2 + t \\ z = 1 + t \end{cases} \]

\[ \begin{align*}
2 - a &= 4 - b \\
3a &= 2 + b \\
1 + 2a &= 1 + b
\end{align*} \]

Pick two to solve:

2 - a = 4 - b \\
+ 3a = 2 + b \\
\underline{2 + 2a = 6} \\
2a = 4 \\
\underline{a = 2}

Test in third eq’n:

1 + 2 \cdot 2 = 1 + 4 \\
5 = 5 \checkmark \quad \text{The lines intersect!}

\[ \begin{align*}
x &= 2 - 2 = 0 \\
y &= 3 \cdot 2 = 6 \\
z &= 1 + 2 \cdot 2 = 5
\end{align*} \]

Point of intersection: \((0, 6, 5)\)
5. (10 points) Match the equation with the surface. Write the letter on the line provided next to each equation.

\[ z = x^2 + y^2 \]  \[ x^2 + y^2 - z^2 = -1 \]  \[ x^2 + y^2 - z^2 = 1 \]  \[ x^2 + y^2 - z^2 = 0 \]  \[ z = y^2 - x^2 \]

(A) \[ z = x^2 + y^2 \]  \[ x^2 + y^2 - z^2 = -1 \]  \[ x^2 + y^2 - z^2 = 1 \]  \[ x^2 + y^2 - z^2 = 0 \]  \[ z = y^2 - x^2 \]
6. (5 points) Which of the following is an equation of the sphere that is centered at \(P(1, 2, 3)\) and is tangent to the \(xz\)-plane?
   \(a\) \((x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 1\)
   \(b\) \((x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 4\)
   \(c\) \((x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 9\)
   \(d\) \((x + 1)^2 + (y + 2)^2 + (z + 3)^2 = 4\)
   \(e\) \((x + 1)^2 + (y + 2)^2 + (z + 3)^2 = 9\)

7. (5 points) Consider the planes \(P_1\) and \(P_2\), shown below.
\[
\begin{align*}
P_1 &: x + 2y + 3z = 7 \\
P_2 &: 2x - 4y + kz = 11
\end{align*}
\]
For which value of \(k\) are the planes \(P_1\) and \(P_2\) perpendicular?
   \(a\) \(k = -2\)
   \(b\) \(k = -1\)
   \(c\) \(k = 0\)
   \(d\) \(k = 1\)
   \(e\) \(k = 2\)

8. (5 points) Consider the vectors \(\vec{v} = \langle 1, -2, -1 \rangle\) and \(\vec{b} = \langle -2, 1, 2 \rangle\). Which of the following is the projection of \(\vec{v}\) onto \(\vec{b}\)?
   \(a\) \(\langle \frac{4}{3}, \frac{2}{3}, -\frac{4}{3} \rangle\)
   \(b\) \(\langle -\frac{2}{3}, \frac{4}{3}, -\frac{2}{3} \rangle\)
   \(c\) \(\langle 4, -2, -4 \rangle\)
   \(d\) \(\langle -2, 4, 2 \rangle\)
   \(e\) \(\langle 2, -4, -2 \rangle\)
9. (5 points) Which of the following is the domain of the function \( f(x, y) = \ln(x^2 + y^2 - 4) \)? Note that for (A), (C), and (E) boundary is dotted.
10. (5 points) Use the following graph to answer the next two questions.

Which of the following is \( \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} \)?

(a) \( (4, 0) \)  
(b) \( (0, 4) \)  
(c) \( (6, 1) \)  
(d) \( (1, 6) \)  
(e) \( (-1, 6) \)

11. (5 points) Which of the following is the vector \( \overrightarrow{BC} - \overrightarrow{CD} \)?

(a) \( (0, 2) \)  
(b) \( (-1, -1) \)  
(c) \( (1, 1) \)  
(d) \( (2, 0) \)  
(e) \( (1, -1) \)