GOALS

▸ Be able to compute a gradient vector, and use it to compute a directional derivative of a given function in a given direction.

▸ Be able to use the fact that the gradient of a function $f(x,y)$ is perpendicular (normal) to the level curves $f(x,y)=k$ and that it points in the direction in which $f(x,y)$ is increasing most rapidly.
INTRODUCTION

- So far, we’ve been able to figure out the rate of change of a function of two variables in only two directions: the $x$-direction or the $y$-direction.

- Now, what if we want to find the rate of change of a function in any other direction?

  - How do we point to things in 3-Space?

    - A: Vectors

- So the question is this: **How do we compute the rate of change of a function in the direction of a given vector at a given point?**
Consider a surface given by $z = f(x, y)$

Say we want to find the rate at which $f$ is changing at the point $(x_0, y_0)$ in the direction of a unit vector $\langle a, b \rangle$

We can draw a line through $(x_0, y_0)$ in the direction of $\langle a, b \rangle$

And then a plane coming straight up from the line

The trace we get on that plane is what we’re looking for
The line on the xy-plane through \((x_0, y_0)\) in the direction of \(<a, b>\):

\[
l : \begin{cases} 
  x(t) = x_0 + at \\
  y(t) = y_0 + bt 
\end{cases}
\]

To get the trace, we want to plug only the points on the line into the function \(f\):

\[
z = f(x(t), y(t))
\]

The slope we want is the derivative of \(z\) with respect to \(t\):

\[
\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
\]

\[
= \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b
\]
THE DIRECTIONAL DERIVATIVE

The directional derivative of the function $f(x,y)$ at the point $(x_0,y_0)$ in the direction of the unit vector $<a,b>$ is written and computed as follows:

$$D_{\mathbf{u}} f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)a + \frac{\partial f}{\partial y}(x_0, y_0)b$$

We can write this as a dot product:

$$D_{\mathbf{u}} f(x_0, y_0) = \left\langle \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right\rangle \cdot <a, b>$$

We have unit vector $\mathbf{u}=<a,b>$ and the vector with the first order partial derivatives. We call this second vector the **gradient**.
We write and define the gradient as follows:

\[ \nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle f_x, f_y \right\rangle \]

\[ \nabla F(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \left\langle f_x, f_y, f_z \right\rangle \]

E.g. Given \( f(x, y) = 3x^2y^3 \), the gradient is

\[ \nabla f(x, y) = \left\langle 6xy^3, 9x^2y^2 \right\rangle \]
PUTTING IT ALL TOGETHER

- The directional derivative of the function $f(x,y)$ at the point $(x_0,y_0)$ in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ is given by

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

- E.g.

$$f(x, y) = x^3 - 2xy^2$$

$$(x_0, y_0) = (1, -1)$$

$$\vec{u} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$$

- For the gradient, we have

$$\nabla f = \langle 3x^2 - 2y^2, -4xy \rangle$$

- Evaluate the gradient at the given point

$$\nabla f(1, -1) = \langle 1, 4 \rangle$$

- Plugging everything in we get...

$$D_{\vec{u}} f(1, -1) = \langle 1, 4 \rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$= -\frac{1}{\sqrt{2}} + \frac{4}{\sqrt{2}}$$

$$= \frac{3}{\sqrt{2}}$$
EXAMPLE

- Consider the function
  \[ f(x, y) = e^y \sin(x) \]

- Compute the directional derivative of \( f \) at \((\pi/6, 0)\) in the direction of the vector \(<2, 3>\)

- Find a set of parametric equations for the tangent line whose slope you found above

\[ D_{\vec{u}} f(x_0, y_0) = \overrightarrow{\nabla} f(x_0, y_0) \cdot \vec{u} \]
We need a unit vector, so we have to normalize \( \langle 2, 3 \rangle \):

\[
\| \langle 2, 3 \rangle \| = \sqrt{13} \quad \vec{u} = \frac{1}{\sqrt{13}} \langle 2, 3 \rangle
\]

The gradient of \( f \) is

\[
\nabla f = \langle e^y \cos x, e^y \sin x \rangle
\]

\[
\nabla f(\pi/6, 0) = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle
\]

To make the dot product easier to compute, we can factor out the fractions

\[
D_{\vec{u}} f(\pi/6, 0) = \frac{1}{2\sqrt{13}} \left( \langle 2, 3 \rangle \cdot \langle \sqrt{3}, 1 \rangle \right)
\]
For the tangent line, we need a point (of tangency) and a direction vector.

- The point is \((\pi/6,0,f(\pi/6,0)) = (\pi/6,0,1/2)\)

- The direction vector is \(\left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, \frac{2\sqrt{3} + 3}{2\sqrt{13}} \right\rangle\)

So we have:

\[ l : \begin{align*}
x &= \frac{\pi}{6} + \frac{2}{\sqrt{13}} t \\
y &= \frac{3}{\sqrt{13}} t \\
z &= \frac{1}{2} + \frac{2\sqrt{3} + 3}{2\sqrt{13}} t
\end{align*} \]

The first two components are the unit vector \(u\)

\[ l : \begin{align*}
x &= \frac{\pi}{6} + 4t \\
y &= 6t \\
z &= \frac{1}{2} + (2\sqrt{3} + 3)t
\end{align*} \]
PROPERTIES OF THE GRADIENT

Let’s take a closer look at the directional derivative:

\[ D_{\bar{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \bar{u} \]

The gradient points in the direction in which the directional derivative is greatest or at any given point, a function’s gradient points in the direction in which the function increases most rapidly.

It’s a unit vector so its norm is 1!

This quantity is largest when \( \theta \) is 0... or when the gradient is in the same direction as \( \bar{u} \)
EXAMPLE

- Consider the function \( f(x,y) = 4 - x^2 - y^2 \) at the point \((1,1)\).

- We would expect the directional derivative to be greatest when walking toward the z-axis.

- Compute the gradient at \((1,1)\):

\[
\nabla f = \langle -2x, -2y \rangle \\
\n\nabla f(1,1) = \langle -2, -2 \rangle
\]

- It works! This vector points directly at the z-axis from \((1,1)\).
ONE MORE GRADIENT PROPERTY

- Consider a level curve $f(x,y) = k$ which contains the point $(x_0, y_0)$. We could represent this curve as a vector-valued function...

  ![Diagram of a level curve and gradient vector]

  \[ \mathbf{r}(t) = \langle x(t), y(t) \rangle \]

- So we have:

  \[ \frac{d}{dt} \left[ f(x(t), y(t)) \right] = \frac{d}{dx}[k] \]

  **SO THE GRADIENT IS NORMAL TO THE LEVEL CURVE**
**EXAMPLE**

- Consider the function $f(x,y) = x^2 - y^2$
- Draw the level curve of $f$ that contains the point $(2,1)$
- Compute the gradient of $f$ at the point $(2,1)$
- Sketch the level curve and then draw the gradient at the point $(2,1)$

\[
f(2, 1) = 3
\]
\[
x^2 - y^2 = 3
\]

\[
\nabla f = \langle 2x, -2y \rangle
\]
\[
\nabla f(2, 1) = \langle 4, -2 \rangle
\]
ONE MORE EXAMPLE

Consider the function $f(x,y) = x^2 - y$

Compute the directional derivative of $f$ at $(2,3)$ in the direction in which it increases most rapidly

Draw level curves for $f(x,y) = -1, 0, 1, 2, 3$

Draw the gradient of $f$ at $(2,3)$
- $f$ increases most rapidly in the direction of its gradient
  \[ \nabla f(x, y) = \langle 2x, -1 \rangle \]
  \[ \nabla f(2, 3) = \langle 4, -1 \rangle \]

- Normalizing the gradient to get a unit vector, we get
  \[ \frac{\nabla f(2, 3)}{||\nabla f(2, 3)||} = \frac{1}{\sqrt{17}} \langle 4, -1 \rangle \]

- Plugging everything in to get the directional derivative...
  \[ D_{\mathbf{u}} f(2, 3) = \frac{1}{\sqrt{17}} \langle 4, -1 \rangle \cdot \langle 4, -1 \rangle \]

**Notice:** The directional derivative in the direction of the gradient is equal to the magnitude of the gradient.
Some level curves:

- $f = -1$:
  - $-1 = x^2 - y$
  - $y = x^2 + 1$

- $f = 0$:
  - $0 = x^2 - y$
  - $y = x^2$

- $f = 1$:
  - $1 = x^2 - y$
  - $y = x^2 - 1$

- $f = 2$: $y = x^2 - 2$

- $f = 3$: $y = x^2 - 3$

Notice: At the point $(2,3)$, the gradient is normal to the level curve and is pointing in the direction in which $f$ is increasing.