MATH 200
WEEK 1- FRIDAY

DOT PRODUCTS AND PROJECTIONS
MAIN QUESTIONS FOR TODAY

- How is the **dot product** defined for vectors?
- How does it interact with other operations on vectors?
- What uses are there for the dot product?
**DEFINITION**

- The dot product is a new kind of operation in that it takes in two objects of one kind and yields an object of a different kind!

- It takes two vectors and gives a scalar

- Given \( \mathbf{v} = <v_1, v_2, v_3> \) and \( \mathbf{w} = <w_1, w_2, w_3> \), we define the dot product as follows

  \[ \mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3 \]

- E.g. If \( \mathbf{v} = <2, 1, -2> \) and \( \mathbf{w} = <3, -4, -1> \), then

  \[ \mathbf{v} \cdot \mathbf{w} = (2)(3) + (1)(-4) + (-2)(-1) = 6 - 4 + 2 = 4 \]
Compute the following dot products:

\[ \langle 1, 4, 5 \rangle \cdot \langle 2, 2, 1 \rangle \]

\[ (3\hat{i} - 2\hat{k}) \cdot (\hat{i} - 10\hat{j} + \hat{k}) \]
The dot product is called a product because of how it interacts with vector addition:

\[ \vec{a} \cdot (\vec{v} + \vec{w}) = \vec{a} \cdot \vec{v} + \vec{a} \cdot \vec{w} \]

It’s commutative (meaning the order in which we multiply doesn’t matter):

\[ \vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \]

And it can be used to define the norm of a vector more succinctly:

\[ \vec{v} \cdot \vec{v} = ||\vec{v}||^2 \]

***For each property, you should confirm with examples***
WHAT DOES THIS DO FOR US?

▸ Remember of the Law of Cosines…?

▸ Of course you do - it’s a generalized Pythagorean Theorem

\[ c^2 = a^2 + b^2 - 2ab \cos \theta \]
Let’s redraw the law of cosines diagram with vectors instead:

\[ \mathbf{v} - \mathbf{w} \]

\[ c^2 = a^2 + b^2 - 2ab \cos \theta \]

\[ \| \mathbf{v} - \mathbf{w} \|^2 = \| \mathbf{v} \|^2 + \| \mathbf{w} \|^2 - 2 \| \mathbf{v} \| \| \mathbf{w} \| \cos \theta \]
\[ ||\vec{v} - \vec{w}||^2 = ||\vec{v}||^2 + ||\vec{w}||^2 - 2||\vec{v}||||\vec{w}|| \cos \theta \]

**EXPAND THIS TERM**

\[ ||\vec{v} - \vec{w}||^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) \]

**PLUG BACK IN**

\[ ||\vec{v}||^2 - 2\vec{v} \cdot \vec{w} + ||\vec{w}||^2 = ||\vec{v}||^2 + ||\vec{w}||^2 - 2||\vec{v}||||\vec{w}|| \cos \theta \]
QUICK CONCLUSIONS FROM THE DOT PRODUCT

- Say we compute the dot product of two vectors $\mathbf{v}$ and $\mathbf{w}$. The result will be **positive**, **negative**, or **zero**.

- What can we say about the angle between the vectors in each case?
  - If $\mathbf{v} \cdot \mathbf{w} > 0$: $\cos \theta > 0$ so the angle is acute
  - If $\mathbf{v} \cdot \mathbf{w} < 0$: $\cos \theta < 0$ so the angle is obtuse
  - If $\mathbf{v} \cdot \mathbf{w} = 0$: $\cos \theta = 0$ so the angle is $90^\circ$

- We use the word **orthogonal** to refer to vectors that form a $90^\circ$ angle.

Reminder:

$$
\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}|| ||\mathbf{w}||}
$$
PROJECTIONS

- Say we have two vectors $\mathbf{v}$ and $\mathbf{b}$, and we want to do the following:
  - Draw $\mathbf{v}$ and $\mathbf{b}$ tail to tail
    - For the sake of this illustration make $\mathbf{b}$ longer than $\mathbf{v}$ though it doesn’t matter
  - Drop a line that’s perpendicular to $\mathbf{b}$ from the tip of $\mathbf{v}$
  - Find the vectors that form the right triangle that results

THIS VECTOR IS CALLED THE PROJECTION OF $\mathbf{v}$ ONTO $\mathbf{b}$
We write the projection of \( \mathbf{v} \) onto \( \mathbf{b} \) as \( \text{proj}_b \mathbf{v} \).

From the picture it should be clear that
\[
||\text{proj}_b \mathbf{v}|| = ||\mathbf{v}|| \cos \theta
\]

\( \mathbf{b}/||\mathbf{b}|| \) is a unit vector in the direction of \( \mathbf{b} \) so...

\[
\text{proj}_b \mathbf{v} = ||\text{proj}_b \mathbf{v}|| \frac{\mathbf{b}}{||\mathbf{b}||}
\]
Putting it all together...

\[ ||\overrightarrow{\text{proj}_b \vec{v}}|| = ||\vec{v}|| \cos \theta \]

\[ \overrightarrow{\text{proj}_b \vec{v}} = ||\overrightarrow{\text{proj}_b \vec{v}}|| \frac{\vec{b}}{||\vec{b}||} \]

\[ \overrightarrow{\text{proj}_b \vec{v}} = ||\vec{v}|| \frac{\vec{b}}{||\vec{b}||} \cos \theta \]

\[ \overrightarrow{\text{proj}_b \vec{v}} = ||\vec{v}|| \frac{\vec{b}}{||\vec{b}||} \left( \frac{\vec{v} \cdot \vec{b}}{||\vec{v}|| ||\vec{b}||} \right) \]

\[ \overrightarrow{\text{proj}_b \vec{v}} = \left( \frac{\vec{v} \cdot \vec{b}}{||\vec{b}||^2} \right) \vec{b} \]
DISTANCE FROM A POINT TO A LINE

- Let’s use projections to find the distance from a point to a line.
  - Find the (shortest) distance from the point $A(3,1,-1)$ to the line containing $P_1(6,3,0)$ and $P_2(0,3,3)$
- We’re all about vectors now so let’s draw some...