PARAMETRIC EQUATIONS OF LINES
MAIN QUESTIONS FOR TODAY

- How do we describe lines in space?
- How do we determine if two lines are parallel, intersecting, or skew?
- How do we figure out where lines intersect the coordinate planes?
DESCRIBING THE POINTS ON A PARTICULAR LINE

- A line is uniquely defined by **two points** (just like in 2D)
- We can draw a bunch of vectors between these points and the origin
- We can use these vectors to come up with a set of equations for any point on the line...

\[ \mathbf{v} = \mathbf{P}_2 - \mathbf{P}_1 = \langle x_2-x_1, y_2-y_1, z_2-z_1 \rangle \]

But we'll just write \( \langle a, b, c \rangle \) for short
We want to describe an arbitrary point on the line in terms of the vectors we drew.

- \( <x_1, y_1, z_1> \) will get us to \( P_1 \)
- \( <a, b, c> \) gets us from \( P_1 \) to \( P_2 \)
- Scalar multiples of \( <a, b, c> \) will go back and forth along the line!
What’s the relationship between the vector that points directly to \(\langle x, y, z \rangle\) and the other vectors?

\[
\langle x, y, z \rangle = \langle x_1, y_1, z_1 \rangle + t \langle a, b, c \rangle
\]

\[
\langle x, y, z \rangle = \langle x_1 + at, y_1 + bt, z_1 + ct \rangle
\]

\[
\begin{align*}
x &= x_1 + at \\
y &= y_1 + bt \\
z &= z_1 + ct
\end{align*}
\]
EXAMPLE 1

- Compute a set of parametric equations for the line containing the points A(1,3,4) and B(-2,1,7)

First, we find a vector parallel to the line

\[ \mathbf{v} = <-2 - 1, 1 - 3, 7 - 4> = <-3, -2, 3> \]

Now we pick a starting point

Either A or B will do

\[
\begin{cases}
  x = 1 - 3t \\
  y = 3 - 2t \\
  z = 4 + 3t
\end{cases}
\]
YOU TRY ONE

- Find a set of parametric equations for the line containing the points A(4,-2,0) and B(3,3,-1).
- Find another point on the line.

\[
\begin{align*}
x &= 4 - t \\
y &= -2 + 5t \\
z &= -t
\end{align*}
\]

To get another point on the line, we can just plug in another “time” value for \( t \).

E.g. When \( t = -1 \) we get (5, -7, 1)
ANOTHER EXAMPLE

▸ Find a set of parametric equations for the line that passes through the point $A(1,4,2)$ and is parallel to the $xy$-plane and the $yz$-plane.

▸ If it’s parallel to both the $xy$-plane and the $yz$-plane, then its direction vector must have $x$- and $z$-components that are zero.

▸ e.g. $<0,1,0>$ would work

\[
\begin{align*}
    x &= 1 \\
    y &= 1 + t \\
    z &= 2
\end{align*}
\]
PARALLEL, INTERSECTING, OR SKEW

- Parallel
  - Parallel lines have parallel direction vectors
  - e.g.
    \[
    \begin{align*}
    x &= 3 - 2t \\
    y &= 1 + t \\
    z &= 2 + t
    \end{align*}
    \]
    \[
    \begin{align*}
    x &= 4t \\
    y &= 5 - 2t \\
    z &= 9 - 2t
    \end{align*}
    \]
  - The first line is parallel to the vector \(-2, 1, 1\)
  - The second line is parallel to \(<4, -2, -2>\)
  - Since, \(-2<-2, 1, 1> = <4, -2-2>\), the lines are parallel.
Skew vs intersecting

Suppose we have two lines that are not parallel

e.g.

\[ L_1 : \begin{cases} 
    x = 1 - t \\
    y = 2 + 3t \\
    z = -1 - 2t 
\end{cases} \quad \begin{cases} 
    x = 3 + 4t \\
    y = 1 - t \\
    z = t 
\end{cases} \]

We know they’re not parallel because \(-1,3,-2\) is not parallel to \(4,-1,1\).

They **may** intersect, but they may pass right by one another...
Even if they do intersect, they may pass through their common point for different $t$-values!

$L_1: \begin{cases} x = 1 - t \\ y = 2 + 3t \\ z = -1 - 2t \end{cases} \quad L_2: \begin{cases} x = 3 + 4t \\ y = 1 - t \\ z = t \end{cases}$

Rather than setting the equations equal as given, we’ll need different letters for the parameter in each!

$1 - a = 3 + 4b$

$2 + 3a = 1 - b$

$-1 - 2a = b$

Pick two equations and solve for $a$ and $b$