Section 13.8: Relative \& Absolute Extrema for Functions of Two Variables

From Calc I:

Relative Extrema:

1. Find critical pts. i.e. pts where \( f' = 0 \) or \( f' \) is undefined.

2. Use 1st or 2nd Deriv. Test to determine if max or min.

Absolute Extrema (on a closed interval):

1. Find critical points.

2. Evaluate \( f \) at endpoints.

3. Compare values from 1 \& 2.

Open \& Closed Regions:

Open: Does not include the boundary.

Closed: Includes boundary.

Critical Points: \((x_c, y_c)\)

\[
\begin{align*}
&f_x(x_c, y_c) = 0 \\
&f_y(x_c, y_c) = 0
\end{align*}
\]
2nd Partial Test:

\[ D(x,y) = \frac{\partial^2}{\partial x \partial y}(x,y) - \frac{\partial^2}{\partial y \partial x}(x,y) \]

\[ = \begin{vmatrix} \frac{\partial^2}{\partial x \partial x} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial y \partial x} & \frac{\partial^2}{\partial y \partial y} \end{vmatrix} \]

If...

1. \( D(x_c, y_c) > 0 \) and \( \frac{\partial^2}{\partial x \partial y}(x_c, y_c) > 0 \) (or \( \frac{\partial^2}{\partial y \partial y}(x_c, y_c) < 0 \))
   THEN REL. MIN AT \((x_c, y_c)\)

2. \( D(x_c, y_c) > 0 \) and \( \frac{\partial^2}{\partial x \partial y}(x_c, y_c) < 0 \) (or \( \frac{\partial^2}{\partial y \partial y}(x_c, y_c) > 0 \))
   THEN REL. MAX AT \((x_c, y_c)\)

3. \( D(x_c, y_c) < 0 \), THEN \((x_c, y_c)\) IS A SADDLE POINT

4. \( D(x_c, y_c) = 0 \), THE TEST IS INCONCLUSIVE

*Note: The reason we can look at \( \frac{\partial^2}{\partial x \partial y} \) or \( \frac{\partial^2}{\partial y \partial y} \) in cases 1 & 2:

\[ \left[ \frac{\partial^2}{\partial x \partial y}(x,y) \right]^2 > 0 \] (Any real \( \neq 0 \) is pos.)

If \( D > 0 \), \( \frac{\partial^2}{\partial x \partial y} \) \( \frac{\partial^2}{\partial y \partial y} \) MUST BE POSITIVE.

\( \frac{\partial^2}{\partial x \partial x} \) \( \frac{\partial^2}{\partial y \partial y} \) IS POSITIVE IF \( \frac{\partial^2}{\partial x \partial y} \) \( \frac{\partial^2}{\partial y \partial y} \) HAVE SAME SIGN

Ex: \( f(x,y) = x^2 + y^2 - 3x - 4y + 6 \)

FIND CRITICAL POINTS
\[ \frac{p_x}{x} = 2x - 3 \]
\[ 2x - 3 = 0 \]
\[ x = \frac{3}{2} \]

\[ \frac{p_y}{y} = 2y - 4 \]
\[ 2y - 4 = 0 \]
\[ y = 2 \]

\[ (\frac{3}{2}, 2) \]

- **Evaluate** \( D(\frac{3}{2}, 2) \)
  
  \[ \frac{p_{xx}}{x} = 2 \]
  
  \[ \frac{p_{yy}}{y} = 2 \]
  
  \[ \frac{p_{xy}}{xy} = 0 \]

  \[ D(\frac{3}{2}, 2) = 2(2) - 0 = 4 > 0 \]

  \[ \frac{p_{xx}}{x}(\frac{3}{2}, 2) = 2 > 0 \]

  **Relative min at** \( (\frac{3}{2}, 2) \)

- **Ex: \( f(x, y) = x^2y - 6y^2 - 3x^2 \)**

- **Critical Points: \( \frac{p_x}{x} = 2xy - 6x \)**

  \[ 2xy - 6x = 0 \]

  \[ 2x(y - 3) = 0 \]

  \[ x = 0, y = 3 \]

- **Critical Points: \( \frac{p_y}{y} = x^2 - 12y \)**

  \[ x^2 - 12y = 0 \]

  \[ \text{If } x = 0; -12y = 0 \]

  \[ y = 0 \]

  \[ \text{If } y = 3; x^2 - 12(3) = 0 \]

  \[ x^2 = 36 \]

  \[ x = \pm 6 \]

- **Critical Points:** \( (0, 0), (6, 3), (-6, 3) \)

- **2nd Partial Test:**
  
  \[ \frac{p_{xx}}{x} = 2y - 6 \]

  \[ \frac{p_{yy}}{y} = -12 \]

  \[ \frac{p_{xy}}{xy} = 2x \]

  **Notice:** \( \frac{p_{yx}}{yx} = 2x \) too
\begin{align*}
(0,0): & \quad D(0,0) = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}^2(0,0) \\
& = (-6)(-12) - 0 = 72 > 0 \\
& \Rightarrow f_{xx}(0,0) = -6 < 0 \\
& \text{RELATIVE MAX AT } (0,0) \\
(6,3): & \quad D(6,3) = (6)(-12) - (2(6))^2 \\
& = -144 < 0 \\
& \text{SADDLE POINT AT } (6,3) \\
(-6,3): & \quad D(-6,3) = (6)(-12) - (2(-6))^2 \\
& = -144 < 0 \\
& \text{SADDLE POINT AT } (-6,3) \\

*GRAPH OF \ z = \frac{1}{10} f(x,y) *
\end{align*}
Finding Absolute Extrema:

**Ex:** Find abs. extrema of \( f(x,y) = x^2 y^2 \) on \( R: \frac{x^2}{4} + \frac{y^2}{9} \leq 1 \)

* abs. extrema will occur at either
  (1) Critical Points inside region
  or (2) Boundary Points

(1) \( \frac{\partial}{\partial x} = 2x \quad \frac{\partial}{\partial y} = 2y \)

2x = 0 \quad 2y = 0

x = 0 \quad y = 0

Only critical point: (0,0)

(2) Boundary: \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \)

* To see what function values on boundary look like we "plug the boundary into the function".

\[ \frac{x^2}{4} + \frac{y^2}{9} = 1 \implies \frac{y^2}{9} = 1 - \frac{x^2}{4} \]

\[ y^2 = 9 - \frac{9x^2}{4} \]

\[ f \text{ on Boundary: } f(x) = x^2 + 9 - \frac{9}{4} x^2 \]

\[ = 9 - \frac{5}{4} x^2 \quad [-2, 2] \]

*This is a Calc I abs. ext. problem*

Critical Points: \( f'(x) = -\frac{5}{2} x \)

- \( \frac{5}{2} x = 0 \)

- \( x = 0 \)

Note: When \( x = 0 \) on boundary \( y^2 = 9 - \frac{9}{4} (0) \)

- \( y^2 = 9 \)

- \( y = 3, -3 \)
TEST VALUES:
\[
\begin{align*}
B(0) &= 9 \\
B(2) &= 4 \\
B(-2) &= 4
\end{align*}
\]

SO EVALUATING \(B(0)\) IS THE SAME AS EVALUATING \(f(0,3)\) OR \(f(0,-3)\).

SIMILARLY: \(B(2) = 4 \implies f(2,0) = 4\)
\(B(-2) = 4 \implies f(-2,0) = 4\)

ABSOLUTE MAX: 9 AT (0, 3) & (0, -3)

ABSOLUTE MIN: 0 AT (0, 0)

\[f(x) = x^2 - 4xy + 5y^2 - 8; \text{ } R: \text{ TRIANGLE W/ VERTICES } (0,0), (3,0), (3,3)\]

(1) CRITICAL POINTS:
\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2x - 4y \\
\frac{\partial f}{\partial y} &= -4x + 10y
\end{align*}
\]
\[
\begin{align*}
2x - 4y &= 0 \\
-4x + 10y &= 0
\end{align*}
\]

*SOLVE AS SYSTEM OF EQS:*

\[
\begin{align*}
2x - 4y &= 0 \\
-4x + 10y &= 0
\end{align*}
\]

\[
\begin{align*}
2y &= 0 \\
y &= 0
\end{align*}
\]

\[
\begin{align*}
2x - 4(0) &= 0 \\
x &= 0
\end{align*}
\]

CRITICAL POINT IS (0,0)

* THIS POINT LIES ON THE BOUNDARY OF R *

(2) RESTRICT \(f\) TO THE BOUNDARY: WE WILL HAVE TO DO THIS IN 3 PIECES

\[
\begin{align*}
y &= x, \text{ } 0 \leq x \leq 3
\end{align*}
\]

\[
\begin{align*}
B(x) &= x^2 - 4x(x) + 5x^2 - 8 \\
&= 2x^2 - 8 \\
B'(x) &= 4x
\end{align*}
\]

CRIT. PT. \(x = 0\)

\[
\begin{align*}
B(0) &= -8 = f(0,0) \\
B(3) &= 10 = f(3,3)
\end{align*}
\]
(i) $y = 0$, $0 \leq x \leq 3$

$b(x) = x^2 - 8$

$b'(x) = 2x$

$2x = 0$

(ii) $x = 3$, $0 \leq y \leq 3$

$\beta(y) = 9 - 12y + 5y^2 - 8$

$\beta(y) = 5y^2 - 12y + 1$

$\beta'(y) = 10y - 12$

$10y - 12 = 0$

$y = \frac{6}{5}$

**Absolute Max**: $10$ at $(3,3)$

**Absolute Min**: $-8$ at $(0,0)$

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Centered Point $x = 0$

$B(0) = -8 = f(0,0)$

$B(3) = 1 = f(3,0)$

$\beta(0) = 1 = \frac{0}{(3,0)}$

$\beta(3) = 10 = \frac{2}{(3,3)}$

$\beta(\frac{6}{5}) = -\frac{31}{5}$