Notes: Section 11.1

Rectangular Coordinates in 3-Space

In 2-space, points in the plane are labeled with ordered pairs \((x, y)\). The values of \(x\) and \(y\) are the signed distances to the \(x\)- and \(y\)-axes respectively. In other words, \(x = \text{"how far over"}\) and \(y = \text{"how far up"}\).

In 3-space, we add the \(z\)-axis to get 3 mutually perpendicular coordinate axes. The result is that points are labeled with ordered triples \((x, y, z)\).

Two Categories:

* We will just use right-handed *

Ex: Plot the point \((0, 1, 2)\)
THE COORDINATE PLANES:

Often, picturing (and drawing) 3-dimensional objects and surfaces can be difficult. One way to simplify matters is to look at the 2-D projections or cross-sections of 3-D objects. Imagine using photos of a person's head from different angles to create a 3-D sculpture.

When using rectangular coordinates, this amounts to look at one pair of axes at a time. A given pair determines a coordinate plane. There are 3 in 3-space:

- $xy$-plane: $(x, y, 0)$
- $yz$-plane: $(0, y, z)$
- $xz$-plane: $(x, 0, z)$

In 2-space, we have 4 quadrants, in 3-space we have 8 octants.

DISTANCE: It turns out that the Pythagorean theorem doesn't just hold in 2-dimensions; it holds for any number of dimensions!
SO THE DISTANCE BETWEEN 2 POINTS
\((x_1, y_1, z_1)\) AND \((x_2, y_2, z_2)\)
IS GIVEN BY
\[
D = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}
\]

**EX:** FIND THE DISTANCE FROM \((1,1,1)\) TO \((-1,-1,-1)\)

\[
D = \sqrt{(2)^2 + (2)^2 + (2)^2} = \sqrt{12}
\]

**SPHERES:** IN 2-D, THE EQUATION \((x-a)^2 + (y-b)^2 = r^2\) IS THE EQUATION FOR A CIRCLE CENTERED AT \((a, b)\) WITH RADIUS \(r\).

IN 3-D, \((x-a)^2 + (y-b)^2 + (z-c)^2 = r^2\) IS THE EQN FOR A SPHERE WITH RADIUS \(r\) CENTERED AT \((a, b, c)\).

**EX:** FIND THE CENTER AND RADIUS OF THE SPHERE
\[x^2 + y^2 + z^2 = 4\]
\(r = 2\), CENTER: \((0, 0, 0)\)

**EX:** FIND THE CENTER AND RADIUS OF THE SPHERE
\[x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0\]
\[x^2 - 2x + y^2 - 4y + z^2 + 8z = -17\]
\[x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 + 8z + 16 = 4\]
\[(x-1)^2 + (y-2)^2 + (z+4)^2 = (2)^2\]
RADIUS: 2 CENTER: \((1, 2, -4)\)
**Theorem:** An equation of the form
\[ x^2 + y^2 + z^2 + Ax + By + Cz + D = 0 \]
represents a sphere, a point, or has no graph.

**Ex:** \[ x^2 + y^2 + z^2 - 2x - 4y + 8z + 21 = 0 \]
\[ (x-1)^2 + (y-2)^2 + (2+4)^2 = 0 \]
The only solution is \((1, 2, -4)\)

**Ex:** \[ x^2 + y^2 + z^2 + 1 = 0 \]
\[ x^2 + y^2 + z^2 = -1 \]
No solution

**Cylindrical Surfaces:**

**Ex:** By now, we're quite familiar with the graph of \( y = x^2 \) in 2-space: a parabola

![Graph of y = x^2](image)

*All the points \((x, y)\) on the parabola satisfy the equation \( y = x^2 \).*

So, what if we consider the equation \( y = x^2 \) in 3-space?

Well, the graph will be the set of all points \((x, y, z)\) that satisfy \( y = x^2 \). Since this equation doesn't restrict \( z \) at all, \( z \) can take on any value. The result is
IN GENERAL: IF AN EQUATION CONTAINS ONLY 2 OF THE VARIABLES $x, y, z$, THEN THE GRAPH IT REPRESENTS IS A CYLINDRICAL SURFACE.

**Ex:** Sketch the graph of $y^2 + z^2 = 1$

**Ex:** Sketch the graph of $z = \cos x$. 

*Notice if you were to look straight down onto the xy-plane you would just see the familiar 2-D parabola.*
**EX 9:** Find the equation of a sphere with \((1, -2, 4)\) and \((3, 4, -12)\) as endpoints of its diameter.

\[ D = \sqrt{(3-1)^2 + (4-(-2))^2 + (-12-4)^2} = \sqrt{104} \]

Midpoint: \(\left( \frac{1}{2}(1+3), \frac{1}{2}(-2+4), \frac{1}{2}(4-12) \right) = (2, 1, 4) \)

\[(x-2)^2 + (y-1)^2 + (z-4)^2 = 104 \]

**EX 11:** Show \((2, 1, 6), (4, 7, 9), \text{ and } (8, 5, 6)\) are vertices of a right triangle.

\[ a = \sqrt{(2-4)^2 + (1-7)^2 + (6-9)^2} = 7 \]

\[ b = \sqrt{(2-8)^2 + (1-5)^2 + (6-6)^2} = 14 \]

\[ c = \sqrt{(4-8)^2 + (7-5)^2 + (9-6)^2} \]

\[ = \sqrt{25} \]

\[ a^2 + b^2 = c^2 \checkmark \]

**EX 15:** Find the equation of the sphere with center \((2, -1, -3)\) and tangent to the (1) \(xy\)-plane, (2) \(xz\)-plane, (3) \(yz\)-plane.

1. \( r = 3 \)
   \[(x-2)^2 + (y+1)^2 + (z+3)^2 = 9 \]

2. \( r = 1 \)
   \[(x-2)^2 + (y+1)^2 + (z+3)^2 = 1 \]

3. \( r = 2 \)
   \[(x-2)^2 + (y+1)^2 + (z+3)^2 = 4 \]
EX 47: BAG WALKS ON A SPHERE

\[ x^2 + y^2 + z^2 + 2x - 2y - 4z - 3 = 0 \]

**How close/far can it get from the origin?**

\[
(x^2 + 2x + 1) + (y^2 - 2y + 1) + (z^2 - 4z + 4) = 3 + 1 + 1 + 4
\]

\[
(x + 1)^2 + (y - 1)^2 + (z - 2)^2 = 9
\]

Radius: 3; Center: (-1, 1, 2)

**Distance of center to origin is** \( \sqrt{6} \) \( (\sqrt{6} < 3) \)

**Max Dist:** 3 + \( \sqrt{6} \) \n
**Min Dist:** 3 - \( \sqrt{6} \)