Covering Random Points in a Unit Ball

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Abstract

Choose random points $X_1, X_2, X_3, \ldots$ independently from a uniform distribution in a unit ball in $\mathbb{R}^m$. Call $X_n$ a dominator iff $\text{distance}(X_n, X_i) \leq 1$ for all $i < n$, i.e. the first $n$ points are all contained in the unit ball that is centered at the $n$'th point $X_n$. We prove that, with probability one, only finitely many of the points are dominators.

For the special case $m = 2$, we consider the unit disk graph $G_n$ determined by $n$ random points $X_1, X_2, \ldots, X_n$ in the unit disk. With asymptotic probability one, $G_n$ has a connected dominating set consisting of just two points.

**keywords and phrases:** stochastic geometry, dominating set, geometric graph, unit ball graph
1 Introduction

Every finite set \( V \) of points in \( \mathbb{R}^m \), determines a **unit ball graph** \( G(V) = (V, E) \) as follows. The vertex set is \( V \), and an undirected edge \( e \in E \) connects vertices \( u, v \in V \) iff the distance \( d_m(u, v) \) is less than one. (Throughout this paper \( d_m(u, v) \) denotes the Euclidean distance between \( u, v \in \mathbb{R}^m \).) If \( u \) and \( v \) are connected by an edge, we say \( u \) and \( v \) are **adjacent** vertices in the graph \( G(V) \).

The case \( m = 2 \) is particularly prominent in applications, and in this case \( G(V) \) is called a **unit disk graph**. Unit disk graphs have been used by many authors as simplified mathematical models for the interconnections between hosts in a wireless network, and random unit disk graphs have been used as stochastic models for these networks. For other examples of applications, see Marchette [7]. A recent survey of random unit ball graphs is Penrose [8].

A **dominating set** in any graph \( G = (V, E) \) is a subset \( C \subseteq V \) such that every vertex \( v \in V \) either is in the set \( C \), or is adjacent to a vertex in \( C \) [5]. We say \( C \) is a **connected dominating set** if \( C \) is a dominating set and the subgraph induced by \( C \) is connected. The following question arose naturally in the context of routing algorithms for certain wireless networks [9],[4],[3]. Suppose \( V \) is a set of \( n \) random points in the unit disk in \( \mathbb{R}^2 \), and let \( G_n = G(V) \) denote the corresponding unit disk graph. How large, typically, is the smallest connected dominating set for \( G_n \)? This question differs from classical problems in geometric probability in that it is the points themselves, rather than the disk, that must be covered. In this paper we show that with asymptotic probability one, the size of the smallest connected dominating set in \( G_n \) is two.

The paper is organized as follows. In Section 2, we prove a general result concerning unit ball graphs formed from random points \( X_1, X_2, X_3, \ldots \) chosen independently and uniformly in the unit ball in \( \mathbb{R}^m \). It follows from this general result that ‘one point does not suffice’: with asymptotic probability one, \( G_n \) does not have a one-point dominating set. In Section 3 we prove a geometric lemma which is required for the proof, in Section 4, of the existence, with high probability, of two-point connected dominating sets in \( G_n \).

2 One Vertex Dominating Sets

Suppose that \( X_1, X_2, X_3, \ldots \) is a sequence of random points chosen independently from a uniform distribution in a unit ball in \( \mathbb{R}^m \). Call \( X_n \) a **dominator** iff \( d_m(X_n, X_i) \leq 1 \) for all \( i < n \), i.e. all \( n \) points are contained in the unit ball that is centered at \( X_n \). Then we prove:

**Theorem 1** With probability one, only finitely many of the points \( X_n \) are dominators.

**Proof.** Let \( A_n \) be the event that \( X_n \) is a dominator. By the Borel-Cantelli lemma, it suffices to prove that \( \sum_{n=1}^{\infty} \Pr(A_n) < \infty \). For positive real numbers \( r \) and positive integers \( m \geq 2 \), let \( V_m(r) \) be the volume of the a ball of radius \( r \).
Let \( L(r) \) denote the volume of the intersection of two unit balls in \( \mathbb{R}^m \) whose centers are at distance \( r \) from each other. If the distance from the point \( X_n \) to the origin is \( r \), then the conditional probability that the \( i \)’th point \( X_i \) is within distance one of \( X_n \) is \( V_m(1) L(r) \). The distance between the origin and the random point \( X_n \) is a random variable with density \( f(r) = \frac{V_m'(r)}{V_m(1)} m r^{m-1} \).

Hence
\[
\Pr(X_n \text{ is a dominator}) = \int_0^1 f(r) \left( \frac{L(r)}{V_m(1)} \right)^{n-1} dr.
\]

We split the integral into two. Let \( \xi = \frac{4(\log n)V_m(1)}{(n-1)V_m(1)} \). Then
\[
\Pr(X_n \text{ is a dominator}) = I_1 + I_2,
\]
where
\[
I_1 = m \int_0^{\xi} r^{m-1} \left( \frac{L(r)}{V_m(1)} \right)^{n-1} dr
\]
and
\[
I_2 = m \int_{\xi}^1 r^{m-1} \left( \frac{L(r)}{V_m(1)} \right)^{n-1} dr.
\]

For the first piece, we use the trivial estimate \( \frac{L(r)}{V_m(1)} \leq 1 \): for \( m \geq 2 \),
\[
I_1 \leq m \int_0^\xi r^{m-1} dr = \xi^m = O\left( n \log^2 n \right).
\]

To estimate \( I_2 \), we use the following “well-known” formula for \( L(r) \):
\[
L(r) = 2 \int_{r/2}^{1} V_{m-1}(\sqrt{1-x^2}) dx = 2 V_{m-1}(1) \int_{r/2}^{1} (1-x^2)^{\frac{m-1}{2}} dx
\]
\[
= \frac{1}{V_m(1)} \int_{r/2}^{1} \left(1 - \frac{r^2}{4} \right)^{\frac{m-1}{2}} dx
\]

It is intuitively obvious that \( L(r) \) is decreasing, and this is easily confirmed by differentiating the right side of (6) to obtain
\[
L'(r) = -V_{m-1}(1) \cdot \left(1 - \frac{r^2}{4} \right)^{\frac{m-1}{2}} \leq 0
\]
for \( 0 \leq r \leq 1 \). By differentiating again, we also obtain
\[
L''(r) = V_{m-1}(1) \cdot \frac{(m-1)r}{4} \left(1 - \frac{r^2}{4} \right)^{\frac{m-3}{2}} \geq 0
\]
for $0 \leq r \leq 1$. Since $L(r) \leq L(\xi)$ for all $r \geq \xi$, and since $f$ is a density function, we have
\[
I_2 \leq \left( \frac{L(\xi)}{V_m(1)} \right)^{n-1} \int_\xi^1 f(r) \, dr \leq \left( \frac{L(\xi)}{V_m(1)} \right)^{n-1} \tag{9}
\]
To estimate the right side of (9), we note that it follows from (7) and (8) that there is some $0 < c_\xi < \xi$ such that
\[
L(\xi) = L(0) + L'(c_\xi)\xi = V_m(1) - V_{m-1}(1) \cdot \left( 1 - \frac{c_\xi^2}{4} \right)^{\frac{m-1}{2}} \cdot \xi \tag{10}
\]
Since $0 < c_\xi < \xi = o(1)$, we have $(1 - \frac{c_\xi^2}{4})^{\frac{m-1}{2}} > \frac{1}{2}$ for all sufficiently large $n$. So it follows from (10) that
\[
L(\xi) \leq V_m(1) - \frac{V_{m-1}(1)\xi}{2} \tag{11}
\]
for all sufficiently large $n$. Putting (11) back into the right side of (9), we get
\[
I_2 \leq \left( 1 - \frac{\xi V_{m-1}(1)}{2V_m(1)} \right)^{n-1} = O\left( \frac{1}{n^2} \right). \tag{12}
\]
Combining our estimates (5) and (12) for $I_1$ and $I_2$ respectively, we conclude that, for some positive constant $c$, and all sufficiently large $n$, $\Pr(A_n) < c \log^2 n$. Hence $\sum_n \Pr(A_n)$ converges.

Now let $G_n^m \equiv G(X_1, X_2, \ldots, X_n)$ denote the unit ball graph in $\mathbb{R}^m$ with random vertex set $V = \{X_1, X_2, \ldots, X_n\}$. (So, in particular, $G_n = G_n^2$.) Let $B_n^m$ denote the event that $G_n^m$ has a one-point dominating set.

**Corollary 2** For all $m \geq 2$ and for all sufficiently large $n$,
\[
\Pr(B_n^m) \leq \frac{c_m \log^2 n}{n}
\]
where $c_m > 0$ is a positive constant which may depend on the dimension $m$ but does not depend on $n$.

**Proof.** For each $n > 0$ and for $1 \leq i \leq n$, let $B_n(X_i)$ denote the event that $X_i$ is a one-point dominating set of $G_n^m$. Then we have, for all sufficiently large $n$,
\[
\Pr(B_n^m) = \Pr(\cup_{i=1}^n B_n(X_i)) \leq \sum_{i=1}^n \Pr(B_n(X_i)) = n \Pr(B_n(X_1)) \leq \frac{c_m \log^2 n}{n}
\]
since $X_1, X_2, \ldots, X_n$ are independent and identically distributed. We note that the last inequality follows from the bound obtained for $\Pr(A_n)$ in the proof of Theorem 1. \qed
3 A Geometric Lemma

In this section and the next, we adopt the following notation. For any \( r > 0 \), and any \( v \in \mathbb{R}^2 \), let \( D_r(v) \) be the open unit disk centered at \( v \), and let \( \partial D_r(v) \) be the circle of radius \( r \) that bounds it. As observed in [6], a unit disk centered at a point \( o \) cannot be completely covered with two unit disks having centers at points other than \( o \): \( D_1(o) \not\subseteq D_1(u) \cup D_1(v) \) for \( u \neq o \neq v \). The purpose of this section is to prove Lemma 3, which provides an upper bound for the uncovered region’s area (when \( u \) and \( v \) are suitably situated).

Some notation is needed to state Lemma 3. Throughout this section, \( b \geq 3 \). Let \( o = (0, 0) \) be the origin in \( \mathbb{R}^2 \). We are essentially\(^1\) going to partition \( D_b(o) \) into \( 2L_b \) sectors as follows. For integers \( i \) such that \( 0 \leq i < L_b \), let \( Q_i \) be the sector consisting of those points \( (x, y) = (r \cos \theta, r \sin \theta) \) whose polar coordinates satisfy \( 0 < r \leq \delta \) and \( (i - \frac{1}{2}) \theta_b \leq \theta \leq (i + \frac{1}{2}) \theta_b \). Similarly let \( R_i \) consist of the points with \( 0 < r \leq \delta \) and \( (i - \frac{1}{2}) \theta_b \leq \theta - \pi \leq (i + \frac{1}{2}) \theta_b \). Note that the sectors \( Q_i \) and \( R_i \) are located symmetrically with respect to \( o \). Let \( \tilde{q}_i \) and \( \tilde{u}_i \) be the extreme points whose polar coordinates are respectively \( (\delta, (i - \frac{1}{2}) \theta_b) \) and \( (\delta, (i + \frac{1}{2}) \theta_b + \pi) \). Finally, for any points \( u, w \in D_1(o) \), let \( X(u, w) \) denote the area of \( (D_1(u) \cup D_1(w))^c \cap D_1(o) \), i.e. the area of the region in \( D_1(o) \) that is not covered by \( D_1(u) \cup D_1(w) \). Our goal in this section is to prove

**Lemma 3** There is a uniform constant \( C > 0 \) (independent of the parameter \( b \)) such that, for all \( i \), and for all \( q_i, u_i \in R_i \), we have \( X(q_i, u_i) \leq X(\tilde{q}_i, \tilde{u}_i) \leq \frac{C}{b \log b} \).

We prove four facts which together imply Lemma 3. In the first fact, we observe that for any \( v, w \in D_1(o) \) the omitted area \( X(v, w) \) increases if we move one (or both) of the two points \( v \) and \( w \) away from the origin along a radial line.

**Fact 1** Let \( v_1, v_2 \) and \( w_1, w_2 \) be four points in \( D_1(o) \) such that \( v_1 \) lies on the line segment \( \overline{v_2, w_2} \) and \( w_1 \) lies on the line segment \( \overline{v_1, w_2} \). Then \( X(v_2, w_2) > X(v_1, w_1) \).

**Proof.** It suffices to show that \( D_1(v_2) \cap D_1(o) \subseteq D_1(v_1) \cap D_1(o) \) and that \( D_1(w_2) \cap D_1(o) \subseteq D_1(w_1) \cap D_1(o) \). Suppose \( p \in D_1(v_2) \cap D_1(o) \). Since \( v_1 \) lies on the line segment from \( o \) to \( v_2 \), we have \( d(v_1, p) \leq \max(d(o, p), d(v_2, p)) \leq 1 \). Hence \( p \in D_1(v_1) \cap D_1(o) \). By a similar same argument, \( D_1(w_2) \cap D_1(o) \subseteq D_1(w_1) \cap D_1(o) \).

\( \square \)

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\(^1\)It is not strictly correct to call this a partition of \( D_b(o) \) since the origin is omitted, the bounding circle is included, and some pairs of sectors have a non-empty intersection (with zero area).
Fact 2 Let $a, b$ be the two points where the circles $\partial D_1(p), \partial D_1(q)$ intersect. Then, $a, b \perp \overline{p, q}$, and the two line segments $a, b$ and $\overline{p, q}$ intersect at their midpoints.

Proof. This follows immediately from the fact that $d(p, a) = d(p, b) = d(q, a) = d(q, b) = 1$. □

Fact 3 Let $o_1, o_2$ be two points on the circle $x^2 + y^2 = \delta^2_b$. Then, $X(o_1, o_2)$ is a decreasing function of $\angle o_1 oo_2$.

Proof. For convenience, we will use polar coordinates. Without loss of generality, let $o_1$ be the point with polar coordinates $(r_{o_1}, \phi_{o_1}) = (\delta, \pi)$. Let $o_2$ be an arbitrary point on the circle with the polar coordinates $(\delta, \phi_2)$. By symmetry, we only need to consider the case when $o_2$ is in the first or second quadrant; we may, without loss of generality, assume that $0 \leq \phi_2 \leq \pi$. We will show that $X(o_1, o_2)$ is an increasing function of $\phi_2$, then the result follows from the fact that $\angle o_1 oo_2 = \pi - \phi_2$.

Let $a_1, b_1$ be the two points where the circles $\partial D_1(o_1)$ and $\partial D_1(o)$ intersect, with $a_1$ in the second quadrant and $b_1$ in the third quadrant.

Let $o^* \neq o_1$ be a point on the circle $x^2 + y^2 = \delta^2_b$ so that $\partial D_1(o^*)$ meets with both $\partial D_1(o)$ and $\partial D_1(o_1)$ at $a_1$. Let $b^*, d^*$ be the other intersection points of $\partial D_1(o^*)$ with $\partial D_1(o)$ and $\partial D_1(o_1)$, respectively. For convenience, let’s denote $\phi_{o^*}$ by $\phi^*$. Figure 1 illustrates the position of $\partial D_1(o_1), \partial D(o), \partial D_1(o^*)$ and their intersections.

![Figure 1: The position of the circle $\partial D_1(o^*)$](image)

As in the proof of Fact 2, we have $\overline{a_1, d^*} \perp \overline{o_1, o^*}, \overline{a_1, b^*} \perp \overline{o, o^*}$. Notice also
that \( o \) is on the line segment \( a_1, d^* \). So,

\[
\angle b^*a_1o = \angle oo^*o_1 = \angle o^*o_1o = \frac{\phi^*}{2}. \tag{13}
\]

It follows that

\[
0 < \phi^*/2 < \pi/2, \text{ and } \sin \frac{\phi^*}{2} = \frac{\delta_b}{2} \tag{14}
\]

Now, for the point \( o_2 \) with polar coordinates \( (\delta_b, \phi_2) \), let \( a_2, b_2 \) denote the two points where \( \partial D_1(o_2) \) and \( \partial D_1(o) \) intersect, and let \( c_2, d_2 \) denote the two points where \( \partial D_1(o_2) \) and \( \partial D_1(o_1) \) intersect. There are two cases to consider: \( \phi_2 \leq \phi^* \), and \( \phi_2 \geq \phi^* \).

Case 1. \( \phi_2 \leq \phi^* \).

Notice that \( a_1, b_1 \) partitions the circle \( \partial D_1(o) \) into two arcs: the right section and the left section. When, \( \phi_2 \leq \phi^* \), as illustrated in Figure 2, \( a_2, b_2 \) are both on the right section of the circle \( \partial D_1(o) \) between \( a_1, b_1 \). Similarly, \( c_2, d_2 \) are both on the right section of the circle \( \partial D_1(o_1) \) between \( a_1, b_1 \). Clearly,

\[
X(o_1, o_2) = B_1 - (B_2 - B_3) = B_1 - B_2 + B_3,
\]

where

- \( B_1 = \text{area}(D_1(o_1)^c \cap D_1(o)) \)
• $B_2 = \text{area}(D_1(o) \cap D_1(o_2))$

• $B_3 = \text{area}(D_1(o_1) \cap D_1(o_2))$, the shaded area in Figure 2

Notice that $B_3$ is the only area that depends on $\phi_2$. We shall now give an expression for $B_3$. We denote $\angle c_2 o_1 o_2 = y$. Since $\angle o_2 o_1 o = \frac{\phi_2}{2}$, we have

$$0 < y < \frac{\pi}{2}, \text{ and, } \cos y = \delta_b \cos \frac{\phi_2}{2} \quad (15)$$

By symmetry, one can see that the shaded region is partitioned equally by the line $c_2, d_2$. So,

$$B_3 = 2(\frac{2y}{2\pi} \pi - \frac{1}{2}(2\sin y)(\cos y)) = 2y - \sin 2y.$$

Here, the first term is the area of the sector $D_1(o_1)$ that extends from $c_2$ to $d_2$, and the second term is the area of the triangle$(c_2, o_1, d_2)$. From the above two equations, we have

$$\frac{dX(o_1, o_2)}{d\phi_2} = \frac{dB_3}{dy} = \frac{dB_3}{d\phi_2} \cdot \frac{dy}{d\phi_2} = (1 - \cos 2y) \cdot \delta_b \sin \frac{\phi_2}{2} > 0.$$

Here the last inequality follows from the fact that $0 < \frac{\phi_2}{2}, y < \frac{\pi}{2}$. Thus $X(o_1, o_2)$ is an increasing function in $\phi_2$.

Case 2. $\phi_2 > \phi^*$. One can see from Figure 3 that

$$X(o_1, o_2) = B_1 - (B_2 - B_3) = B_1 - B_2 + B_3$$

Where $B_1, B_2$ are defined the same as those in the case 1, but

$$B_3 = \text{area}(D_1(o_1) \cap D_1(o_2) \cap D_1(o))$$

the shaded area in Figure 3

Again, $B_3$ is the only area that depends on $\phi_2$. We will now give an expression for $B_3$. Note that by symmetry, split in half by the line segment $B_3$ is split in half by the line segment $c_2, d_2$. From Figure 1, one can see that

$$\phi_{c_1} = \phi^* + (\frac{\pi}{2} - \angle b^* a_1 o) = \phi^* + (\frac{\pi}{2} - \frac{\phi^*}{2}) = \frac{\pi}{2} + \frac{\phi^*}{2} \quad (16)$$

Now, using the fact that $c_2, d \perp o_1, o_2$, we have

$$\phi_{c_2} = \pi - (\frac{\pi}{2} - \angle o_2 o_1 o) = \pi - (\frac{\pi}{2} - \frac{\phi_2}{2}) = \frac{\pi}{2} + \frac{\phi_2}{2} \quad (17)$$

Using that the circle $\partial D_1(o_1)$ in the polar system is

$$r = \sqrt{1 - \delta_b^2 \sin^2 \phi - \delta_b \cos \phi}$$
Figure 3: The case when $\phi_2 > \phi^*$

and that

$$\phi_{d2} = -(\pi - \phi_{c2}) = -\left(\frac{\pi}{2} - \frac{\phi_2}{2}\right)$$

we get

$$B_3 = 2\left(\int_{\frac{\pi}{2} - \frac{\phi_2}{2}}^{\frac{\pi}{2} + \frac{\phi_2^*}{2}} \int_0^{\sqrt{1 - \delta_b^2 \sin^2 \phi - \delta_b \cos \phi}} r \, dr \, d\phi + \frac{\phi_2}{2 \pi} \cdot \pi\right)$$

$$= \int_{\frac{\pi}{2} - \frac{\phi_2^*}{2}}^{\frac{\pi}{2} + \frac{\phi_2^*}{2}} \left(1 - \delta_b^2 \sin^2 \phi + \delta_b^2 \cos^2 \phi - 2\delta_b \cos \phi \sqrt{1 - \delta_b^2 \sin^2 \phi} \right) d\phi + \frac{\phi_2}{2}$$
Thus,
\[
\frac{dX(\xi, \eta)}{d\phi} = \frac{dB_3}{d\phi} = -\frac{1}{2} \left[ 1 - \delta_b^2 \sin^2\left(-\frac{\phi}{2}\right) + \delta_b^2 \cos^2\left(-\frac{\phi}{2}\right) \right] - 2\delta_b \cos\left(-\frac{\phi}{2}\right) \sqrt{1 - \delta_b^2 \sin^2\left(-\frac{\phi}{2}\right)} + \frac{1}{2} \\
= \frac{1}{2} \left[ \delta_b^2 \cos^2\left(-\frac{\phi}{2}\right) - \delta_b^2 \sin^2\left(-\frac{\phi}{2}\right) + 2\delta_b \sin\left(-\frac{\phi}{2}\right) \sqrt{1 - \delta_b^2 \cos^2\left(-\frac{\phi}{2}\right)} \right] \\
= \frac{1}{2} \left[ \delta_b^2 \cos^2\left(-\frac{\phi}{2}\right) - \delta_b^2 \sin^2\left(-\frac{\phi}{2}\right) \right] + 1 \\
\geq 0
\]
The last inequality follows because \(0 \leq \delta_b \sin\left(-\frac{\phi}{2}\right) \leq 1\), \(0 \leq \sqrt{1 - \delta_b^2 \cos^2\left(-\frac{\phi}{2}\right)} \leq 1\), and thus \((\delta_b \sin\left(-\frac{\phi}{2}\right) - \sqrt{1 - \delta_b^2 \cos^2\left(-\frac{\phi}{2}\right)})^2 \leq 1\).

\[\square\]

**Fact 4** Uniformly for all \(i\), we have \(X(\tilde{q}_i, \tilde{u}_i) = O\left(\frac{1}{b \log^3 b}\right)\).

**Proof.** Without loss of generality, let \(i = 0\) and \(v = (0, 0)\). To simplify notation, define \(x_b = \delta_b \cos\left(-\frac{1}{2} \theta_b\right)\), \(y_b = \delta_b \sin\left(-\frac{1}{2} \theta_b\right)\). Let \((\xi, \eta)\) be the point in the first quadrant where the circles \(x^2 + y^2 = 1\) and \((x - x_b)^2 + (y - y_b)^2 = 1\) meet. Then

\[
X(\tilde{q}_0, \tilde{u}_0) \leq 4 \int_0^\xi \sqrt{1 - x^2} - (y_b + \sqrt{1 - (x - x_b)^2}) dx \\
= -4y_b \xi + 4 \int_0^\xi \frac{-2xx_b + x_b^2}{\sqrt{1 - x^2} + \sqrt{1 - (x - x_b)^2}} dx
\]

Hence we have

\[
X(\tilde{q}_0, \tilde{u}_0) = O(x_b \xi^2) + O(x_b^2 \xi) + O(x_b^3 \xi).
\]

Note that \(x_b^2 + y_b^2 = \delta_b^2 = \frac{1}{\log^2 b}\), that \(\xi^2 + \eta^2 = 1\), that \((\xi - x_b)^2 + (\eta - y_b)^2 = 1\), that \(x_b = \delta_b (1 + O(\theta_b^2))\), and that \(y_b = -\frac{\delta_b \theta_b}{2} (1 + O(\theta_b^2))\). Combining these equations, we get \(\xi = O(\delta_b)\). Putting this estimate back into (19), we get

\[
X(\tilde{q}_0, \tilde{u}_0) = O\left(\frac{1}{b \log^3 b}\right).
\]

\[\square\]

### 4 Two Point Dominating Sets

Let \(n\) be an integer such that \(n \geq 3\), and let \(L_n = \lfloor n^{1/3} (\log n)^2 \rfloor\) and \(\delta_n = \frac{1}{\log^2 n}\). Select \(n\) points \(X_1, X_2, ..., X_n\) independently and uniformly from the unit disk \(D_1(o)\) and form the unit disk graph \(G_n(= G_n^2)\) by putting an
edge between two of the $n$ points iff the distance between them is less than 1.

Our goal in this section is to prove that, with high probability, $G_n$ contains a dominating set consisting of two vertices of $G_n$ that are adjacent to each other.

For $0 \leq i < L_n$, let $Q_i, R_i$ denote the sectors of $D_{\delta_n}(o)$ as defined in the previous section and let $N(Q_i), N(R_i)$ respectively be the number of vertices of $G_n$ that lie in $Q_i$ and $R_i$. Let $\tau_n = \sum_{i=0}^{L_n-1} I_i$ where, in this section only, the indicator variable $I_i = 1$ if and only if $N(R_i) = N(Q_i) = 1$ (and otherwise $I_i = 0$.)

**Lemma 4** $\Pr(\tau_n < n^{1/3} \frac{1}{\log n}) = O(\frac{\log n}{n^{1/3}})$

**Proof.** Let

$$p = \frac{\text{Area}(Q_i)}{\text{Area}(D_1(0))} = \frac{\pi \delta_n^2}{\pi 2L_n} = \frac{1}{2n \log n} \left(1 + O\left(\frac{1}{n^{1/3} \log^2 n}\right)\right).$$

Then

$$E(I_i) = n(n-1)p^2(1 - 2p)^{n-2},$$

and

$$E(\tau_n) = L_n n(n-1)p^2(1 - 2p)^{n-2} = \frac{n^{1/3}}{4(\log n)^6} \left(1 + O\left(\frac{1}{n^{1/3}(\log n)^2}\right)\right).$$

Similary, for $i \neq j$

$$E(I_i I_j) = n(n-1)(n-2)(n-3)p^4(1 - 4p)^{n-4}.$$  \hspace{1cm} (24)

Since $\tau_n = \sum_{i=0}^{L_n-1} I_i$, and the $I_i$’s are identically distributed, we have

$$\text{Var}(\tau_n) = L_n(L_n - 1)E(I_1 I_2) + L_n E(I_1) - (E(\tau))^2.$$  \hspace{1cm} (25)

Combining this identity with the expression for $E(I_i)$ in (22), the expression for $E(I_i I_j)$ in (24), and the definitions for $L_n, \delta_n$ and $p$, we get

$$\text{Var}(\tau_n) = E(\tau_n) \left(1 + O\left(\frac{1}{(\log n)^8}\right)\right).$$

The lemma now follows by Chebyshev’s inequality. \hfill \square

**Theorem 5** There is a constant $c > 0$ such that, with probability greater than $1 - \frac{c}{(\log n)^3}$, the random graph $G_n$ has a connected dominating set that consists of two vertices in $D_{\delta_n}(o)$.

**Proof.**

Let $T_n \subseteq \{0, 1, 2, 3, \ldots, L_n - 1\}$ be the random subset of indices such that $i \in T_n$ iff $N(Q_i) = N(R_i) = 1$. If $T_n \neq \emptyset$, define $Y = \min T_n$ to be the smallest of the indices in $T_n$; otherwise, if $T_n = \emptyset$, set $Y = -1$. Define the indicator
random variable $\chi_n$ as follows: If $\tau_n = |\mathcal{T}_n| = 0$ then $\chi_n = 0$; otherwise, if $\mathcal{T}_n = \{i_1, i_2, \ldots, i_{\tau_n}\}$ and $i_1 < i_2 < \ldots < i_{\tau_n}$, then $\chi_n = 1$ if $Q_{i_1} \cup R_{i_1}$ contains a two-point connected dominating set for $G_n$.

Let $\mathcal{V} = \{X_1, X_2, \ldots, X_n\}$ be the set of vertices of $\mathcal{G}_n$, selected independently and uniformly randomly from $D_1(o)$. Define $Z = \mathcal{V} \cap D_{\delta\cdot}(o)$ to be the set of vertices that lie near the origin $o$, and let $\bar{Z} = |\mathcal{Z}|$ be the number of these points. Then

$$
\Pr(\chi_n = 0) \leq \Pr \left( \chi_n = 0, \tau_n \neq 0, Z \leq \frac{2n^{1/3}}{(\log n)^2} \right) + \Pr(\tau_n = 0)
$$

$$
+ \Pr \left( Z > \frac{2n^{1/3}}{(\log n)^2} \right).
$$

Note that $Z$ has a binomial distribution: $Z \sim Bin(n, \delta_n^2)$. If $\beta = \frac{2n^{1/3}}{(\log n)^2}$, then by Chernoff’s inequality,

$$
\Pr(\chi_n = 0) \leq \Pr(\chi_n = 0, \tau_n \neq 0, Z \leq \beta) + \Theta \left( \frac{\log n}{n^{1/3}} \right).
$$

By Lemma 4, $\Pr(\tau_n = 0) = O\left( \frac{\log n}{n^{1/3}} \right)$. Therefore

$$
\Pr(\chi_n = 0) \leq \Pr(\chi_n = 0, \tau_n \neq 0, Z \leq \beta) + \Theta \left( \frac{\log n}{n^{1/3}} \right).
$$

Now we decompose the first term on the right side of (28) according to the value of $Y$.

$$
\Pr(\chi_n = 0, \tau_n \neq 0, Z \leq \beta) = \sum_{k=0}^{L_n-1} \Pr(\chi_n = 0|Y = k, Z \leq \beta) \Pr(Y = k, Z \leq \beta).
$$

(The redundant condition $\tau_n \neq 0$ need not be included on the right side of (29) because it a consequence of the condition $Y \geq 0$.) We have

$$
\Pr(\chi_n = 0|Y = k, Z \leq \beta) = \sum_{S \subseteq \{n\} \text{ such that } 2 \leq |S| \leq \beta} \Pr(\chi_n = 0|Z = S, Y = k) \Pr(Z = S|Y = k, Z \leq \beta)
$$

where $L_n$ is the number of points that fall outside $D_{\delta\cdot}(o)$, the locations of these $n - |S|$ points are independent of the locations of the $|S|$ points in $D_{\delta\cdot}(o)$. Hence

$$
\Pr(\chi_n = 0|Z = S, Y = k) \geq \frac{1 - \gamma}{\left( 1 - \frac{|D_{\delta\cdot}(o)|}{|D_1(o)|} \right)^{n-|S|}}
$$

where

$$
\gamma = X(q_0, u_0), \quad \text{and recall that } \gamma = O\left( \frac{1}{n \log^2 n} \right).
$$
≥ \left(1 - \frac{C}{n(\log n)^3}\right)^{n-|S|} \geq 1 - \frac{C'}{(\log n)^3} \tag{33}

for some constants C and C' which are independent of Z, Y. Hence

\Pr(\chi_n = 0) \leq \frac{c}{(\log n)^3} \tag{34}

for some positive constant c that does not depend on n.

We note that the result obtained in Theorem 5 depends on a delicate trade-off: We must choose \(\delta_n\) small enough and \(L_n\) large enough to guarantee that for any \(q \in Q_i\) and any \(u \in R_i\), where \((Q_i, R_i)\) is a pair of opposite sectors of \(D_{\delta_n}(o)\), there is high probability that none of the points \(X_1, X_2, ..., X_n\) lie in the ‘uncovered’ region \((D_1(q) \cup D_1(u)') \cap D_1(o)\). On the other hand, \(\delta_n\) must not be so small or \(L_n\) so large that we cannot find (with high probability) some pair of opposite sectors \((Q_i, R_i)\) such that there is some \(X_j \in Q_i\) and \(X_k \in R_i\).

The necessity for this ‘trade-off’ stems from the fact that a unit disk centered at a point \(o\) cannot be completely covered with two unit disks having centers at points other than \(o\), i.e. \(D_1(o) \not\subseteq D_1(u) \cup D_1(v)\) for \(u \neq o \neq v\).

5 Final Comments

The original question posed in the introduction concerned the typical size of a minimum connected dominating set in the random disk graph \(G_n^2\). Theorem 5 establishes that, with asymptotic probability one, \(G_n^2\) has a two-point connected dominating set. By Theorem 1, this two-point dominating set is also a minimum connected dominating set (with asymptotic probability one).

Theorem 5 is difficult because a unit disk, centered at a point \(o\), cannot be completely covered with two unit disks having centers at points other than \(o\). In contrast, one can easily find three points \(u, v, w \in D_1(o) \setminus \{o\}\) such that \(D_1(o) \subseteq D_1(u) \cup D_1(v) \cup D_1(w)\). Using this fact, the authors show in [4] that there is some \(\alpha\), with \(0 < \alpha < 1\), such that for every \(k \geq 3\) the probability that there does not exist a \(k\)-point connected dominating set in \(G_n^2\) is less than \(3\alpha^n\). This exponential probability bound was used to analyze the performance of the Rule \(k\) local algorithm for constructing a connected dominating set in a wireless network model when \(k \geq 3\). By comparing the exponential \(O(\alpha^n)\) probability bound for \(k \geq 3\) with the \(O(\frac{1}{\log n})\) bound for \(k = 2\), we gain some insight into the empirical observation that the Rule \(k\) algorithm does not perform as well for \(k = 2\) as it does for \(k \geq 3\).

Finally, we have not determined the typical size of the minimum connected dominating set for dimensions \(m > 2\). The case \(m = 2\) is already challenging, and we do not see how to extend our methods to the general case. We did prove in [3] that, when \(m = 3\), the probability that there does not exist a 4-point CDS is exponentially small. Therefore, with high probability the smallest CDS in \(G_n^3\) consists of either 2 or 3 vertices. It is reasonable to
conjecture that an analogous statement holds for all $m \geq 2$: with asymptotic probability 1, $G^n_m$ has an $m$ point CDS.

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References


