The Maximum Degree in a Random Tree and Related Problems

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ABSTRACT

Meir and Moon studied the distribution of the maximum degree for simply generated families of trees. We have sharper results for the special case of labelled trees.

1. INTRODUCTION

Recently, Meir and Moon studied the distribution of the maximum degree for simply generated families of trees. In the case of labelled trees, their results imply that the maximum degree is concentrated on an interval of width three. We narrow this interval to two, and give rather explicit expressions for the non-negligible probabilities.

If T is a tree with n labelled vertices, let $D_n(T)$ be the maximum of the degrees of its vertices. One can prove the following theorem using the saddlepoint method (Appendix 1):

Theorem 1. *If* $k_n \sim \dfrac{\log n}{\log\log n}$ *as* $n \to \infty$, *then*

$$\mathrm{Prob}(D_n \le k_n) = \exp[-e^{\log n - k_n \log k_n + k_n - \frac{1}{2}\log k_n - \log(e\sqrt{2\pi}) + o(1)}] + o(1)\,.$$

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Note that, because of the $o(1)$ in the exponent, Theorem 1 does not yield precise asymptotic information unless $\log n - k_n \log k_n + k_n - \frac{1}{2}\log k_n$ is bounded. Nevertheless, this quantity is bounded in the narrow interval that is of interest to us. Hence Theorem 1 can be used to obtain rather explicit formulae for each of the non-negligible probabilities. Let $c = -\log(e\sqrt{2\pi})$, and for positive real numbers t let μ_t be the positive solution to $\log t - x\log x + x - \frac{1}{2}\log x = 0$. Let $\{\mu_t\} := \mu_t - \lfloor \mu_t \rfloor$ be the fractional part of μ_t. Then, by taking $k_n = (\lfloor \mu_n \rfloor - 1)$ in Theorem 1, we get

Corollary 1. *As $n \to \infty$ through the positive integers,*

$$\mathrm{Prob}(D_n \le \lfloor \mu_n \rfloor - 1) = o(1)\,.$$

Next, taking $k_n = \lfloor \mu_n \rfloor$ in Theorem 1, we get

Corollary 2. $\mathrm{Prob}(D_n = \lfloor \mu_n \rfloor) = \exp[-e^{\{\mu_n\}\log\lfloor \mu_n \rfloor + c + o(1)}] + o(1)\,.$

Taking $k_n = \lfloor \mu_n \rfloor + 1$ in Theorem 1, we get

Corollary 3. $\mathrm{Prob}(D_n = \lfloor \mu_n \rfloor + 1) =$

$$\exp[-e^{-(1-\{\mu_n\})\log\lfloor \mu_n \rfloor + c + o(1)}] - \exp[-e^{\{\mu_n\}\log\lfloor \mu_n \rfloor + c + o(1)}] + o(1)\,.$$

Finally, taking $k_n = \lfloor \mu_n \rfloor + 2$, we get $\mathrm{Prob}(D_n \le \lfloor \mu_n \rfloor + 2) = 1 - o(1)$. From this one immediately obtains

Corollary 4. $\mathrm{Prob}(D_n = \lfloor \mu_n \rfloor + 2) = 1 - \exp[-e^{-(1-\{\mu_n\})\log\lfloor \mu_n \rfloor - c - o(1)}] + o(1).$

Now suppose that $\epsilon > 0$ is a fixed, arbitrarily small constant, and suppose that $\langle n_m \rangle_{m=1}^{\infty}$ is a sequence of positive integers such that $\epsilon < \{\mu_{n_m}\} < 1 - \epsilon$. Then, comparing the corollaries to Theorem 1, we get

$$\lim_{m\to\infty} \mathrm{Prob}(D_{n_m} = \lfloor \mu_{n_m} \rfloor + 1) = 1\,.$$

Meir and Moon constructed such subsequences explicitly. Furthermore, because the sequence of fractional parts $\{\mu_n\}$ is dense in $(0, 1)$ (see Corollary 7), there must be many other subsequences that satisfy this condition. It is therefore very tempting to conjecture that D_n is usually concentrated on the single value $\lfloor \mu_{n_m} \rfloor + 1$. More precisely: one might guess that for every $\epsilon > 0$, there is a set of positive integers $A_\epsilon = \{n_1, n_2, n_3 \ldots\}$, with asymptotic density is at least $1 - \epsilon$, such that $n_1 < n_2 < \cdots$ and

$$\lim_{m\to\infty} \mathrm{Prob}(D_{n_m} = \lfloor \mu_{n_m} \rfloor + 1) = 1\,.$$

We were surprised to find that this is NOT correct. In fact, the following is an easy corollary to Theorem 2:

Corollary 5. *For every $\epsilon > 0$, there is a $\delta > 0$ and a set of positive integers $A_\epsilon := \{n_1, n_2, n_3 \ldots\}$, such that $n_1 < n_2 < \cdots$, the asymptotic density of A_ϵ is at least $1 - \epsilon$, and for which one has both*

$$\lim_{m\to\infty} \operatorname{Prob}(D_{n_m} = \lfloor \mu_{n_m} \rfloor + 2) > \delta \,,$$

and

$$\lim_{m\to\infty} \operatorname{Prob}(D_{n_m} = \lfloor \mu_{n_m} \rfloor + 1) > \delta \,.$$

This corollary is of course very crude; Theorem 2 contains considerably more information.

2. CONCENTRATION OF THE MAXIMUM DEGREE

The distribution of the maximum degree is highly concentrated. Meir and Moon's theorem shows that, in a sense, it is concentrated on interval of width less than or equal to three. The corollaries to Theorem 1 enable us to narrow this width to two. Note that the quantities $\{\mu_n\}$ and $(1 - \{\mu_n\})$ cannot *both* be near zero. Therefore, for large n, the three probabilities cannot be simultaneously non-negligible. Thus we have

Corollary 6. *There is a function $m(n)$ such that*

$$\lim_{n\to\infty} \operatorname{Prob}(D_n = m(n) \quad \text{or} \quad D_n = m(n) + 1) = 1 \,.$$

In order to sharpen Corollary 6, we need more information about μ_n and its fractional parts. Because μ_n is slowly increasing, the step function $\lfloor \mu_n \rfloor$ is constant for long intervals. Define N_k to be the kth point of increase, i.e., $N_k := \min\{m\colon \lfloor \mu_m \rfloor = k\}$. Every large integer n lies in exactly one of the intervals $[N_k, N_{k+1})$. Roughly speaking, it is the relative position of n in this interval that determines the magnitude of the three potentially non-negligible probabilities. Before making this all precise, we give an intuitive discussion. Think of x as being the relative position of n in the interval $[N_k, N_{k+1})$. Thus $x = 0$ if $n = N_k$, $x = 1$ if $n = N_{k+1}$, and in general $x \approx \dfrac{n - N_k}{N_{k+1} - N_k}$. We find limit probabilities $P_0(x)$, $P_1(x)$, and $P_2(x)$ such that

$$\operatorname{Prob}(D_n = \lfloor \mu_n \rfloor + d) \approx P_d(x) \,, \quad d = 0, 1, 2 \,. \tag{1}$$

The key to proving this result is the following lemma:

Lemma 1. *If $N_k \le n < N_{k+1}$, then*

$$\{\mu_n\} = \frac{1}{\log k} \log\left(\frac{n}{N_k}\right) + O\left(\frac{1}{k \log k}\right),$$

where the implicit constant is uniform for $N_k \le n < N_{k+1}$.

Proof. We first consider the special case where $n = N_k$. Since $\mu_{N_k-1} < k$, there is an integer between μ_{N_k} and μ_{N_k-1}. Hence

$$\{\mu_{N_k}\} < \mu_{N_k} - \mu_{N_k-1}.$$

By the mean value theorem, there is some $\theta \in (0,1)$ such that the right side is equal to $\left.\frac{d\mu_t}{dt}\right|_{t=N_k-1+\theta}$. On the other hand,

$$\frac{d\mu_t}{dt} = \frac{1}{t}\left(\frac{1}{\log \mu_t + (1/2\mu_t)}\right).$$

For all large t, the function $\log \mu_t + (1/2\mu_t)$ is increasing. Hence

$$\{\mu_{N_k}\} \le \frac{1}{(N_k-1)(\log(\mu_{N_k-1}) + (1/2\mu_{N_k-1})}.$$

Thus

$$\{\mu_{N_k}\} = O\left(\frac{1}{N_k \log \mu_{N_k}}\right). \tag{2}$$

Now we turn to the general case (arbitrary $n \in [N_k, N_{k+1})$). For $t \in [N_k, N_{k+1})$, we have

$$\frac{1}{t}\left(\frac{1}{\log \mu_{N_{k+1}} + (1/2\mu_{N_{k+1}})}\right) \le \frac{d\mu_t}{dt} \le \frac{1}{t}\left(\frac{1}{\log \mu_{N_k} + (1/2\mu_{N_k})}\right).$$

Integrating from N_k to n, we get

$$\frac{\left(\log \frac{n}{N_k}\right)}{\log \mu_{N_{k+1}} + (1/2\mu_{N_{k+1}})} \le \mu_n - \mu_{N_k} \le \frac{\left(\log \frac{n}{N_k}\right)}{\log \mu_{N_k} + (1/2\mu_{N_k})}. \tag{3}$$

Observe that, for $n \in [N_k, N_{k+1})$, $\lfloor \mu_n \rfloor = \lfloor \mu_{N_k} \rfloor = k$, and consequently

$$\mu_n - \mu_{N_k} = \{\mu_n\} - \{\mu_{N_k}\}. \tag{4}$$

It is easy to check that $\log \mu_{N_k} = \log k + O\left(\frac{1}{k}\right)$, and that $\frac{1}{2\mu_{N_k}} = O\left(\frac{1}{k}\right)$. We therefore have

$$\left(\frac{1}{\log \mu_{N_{k+1}} + (1/2\mu_{N_{k+1}})}\right) = \frac{1}{\log k} + O\left(\frac{1}{k \log^2 k}\right). \tag{5}$$

Combining all these estimates, we get

$$\{\mu_n\} - \{\mu_{N_k}\} = \frac{1}{\log k} \log\left(\frac{n}{N_k}\right) + O\left(\frac{\log\left(\frac{n}{N_k}\right)}{k \log^2 k}\right). \tag{6}$$

Using Stirling's formula and some elementary calculations, one can verify that,

$$N_{k+1} = (k+1)N_k + \frac{N_k}{12k} + O\left(\frac{N_k}{k^2}\right), \tag{7}$$

and $N_k \sim k^{k+(1/2)}e^{-k}$ as $k \to \infty$. Combining this with (2) and (6), we get Lemma 1. ∎

As an immediate corollary, one obtains

Corollary 7. *The sequence of fractional parts* $\langle \{\mu_n\} \rangle_{n=1}^{\infty}$ is dense in $(0, 1)$.

Our earlier intuitive discussion of (1) can now be made precise. For $x \in (0, 1)$, let $m = m(x, k) := \lfloor x(N_{k+1} - N_k) \rfloor$, and let

$$P_d^{(k)}(x) := \operatorname{Prob}(D_{N_k+m} = \lfloor \mu_{N_k+m} \rfloor + d), \quad d = 1, 2.$$

For each k, $P_d^{(k)}(x)$ is a step function. But for large k it is approximated by a continuous function:

Theorem 2. *For* $d = 1, 2$, *we have* $P_d^{(k)}(x) \to P_d^{(\infty)}(x)$ *uniformly for* $x \in (0, 1)$ *as* $k \to \infty$, *where*

$$P_1^{(\infty)}(x) = e^{-x/e\sqrt{2\pi}}$$

and

$$P_2^{(\infty)}(x) = 1 - e^{-x/e\sqrt{2\pi}}.$$

Proof. Observe that $\lfloor \mu_{N_k+m} \rfloor = k$. We can therefore use Corollary 4 to conclude that

$$\operatorname{Prob}(D_{N_k+m} = \lfloor \mu_{N_k+m} \rfloor + 2) = 1 - \exp(-e^{-\log k + \{\mu_{N_k+m}\}\log k + c + o(1)}) + o(1) \tag{8}$$

as $k \to \infty$, where the implicit constant is independent of x. We use Lemma 1 to simplify the exponent in (8):

$$\{\mu_{N_k+m}\} = \frac{1}{\log k}\log\left(\frac{N_k+m}{N_k}\right) + O\left(\frac{1}{k\log k}\right),$$

and consequently

$$-\log k + \{\mu_{N_k+m}\}\log k + c = \log\left(\frac{N_k+m}{kN_k}\right) - \log(e\sqrt{2\pi}) + O\left(\frac{1}{k}\right).$$

Recall that $m = \lfloor x(N_{k+1} - N_k) \rfloor$. By (7) we have

$$N_{k+1} - N_k = kN_k + O\left(\frac{N_k}{k}\right).$$

Therefore, in the exponent of (8), we have

$$\begin{aligned}-\log k + \{\mu_{N_k+m}\}\log k + c &= \log\left(\frac{xkN_k + N_k + O(N_k/k)}{e\sqrt{2\pi}kN_k}\right) + O\left(\frac{1}{k}\right)\\ &= \log\left(\frac{x}{e\sqrt{2\pi}} + O\left(\frac{1}{k}\right)\right) + O\left(\frac{1}{k}\right).\end{aligned}$$

Thus

$$\begin{aligned}P_2^{(k)}(x) &= 1 - \exp\left(-\exp\left(\log\left(\frac{x}{e\sqrt{2\pi}} + o(1)\right)(1 + o(1))\right)\right) + o(1)\\ &= 1 - \exp\left(\frac{-x}{e\sqrt{2\pi}} + o(1)\right) + o(1).\end{aligned}$$

By similar arguments,

$$P_1^{(k)}(x) \to \exp\left(\frac{-x}{e\sqrt{2\pi}}\right). \qquad \blacksquare$$

From Theorem 2, we see that $\operatorname{Prob}(D_n = \lfloor\mu_n\rfloor)$ is usually negligible. However, the next theorem shows this probability is non-negligible when n is near the left-hand endpoint of one of the intervals $[N_k, N_{k+1})$.

Theorem 3. *Let $M > 1$ be fixed but arbitrarily large, and let $S := \bigcup_k [N_k, MN_k]$. If $n_1 < n_2 < n_3 < \cdots$ is a sequence in S then*

$$\operatorname{Prob}(D_{n_m} = \lfloor\mu_{n_m}\rfloor) = \exp\left(-\frac{1 + y_{n_m}}{e\sqrt{2\pi}}\right) + o(1),$$

where $y_n := \dfrac{n - N_{\lfloor\mu_n\rfloor}}{N_{\lfloor\mu_n\rfloor}}$.

Proof. By Lemma 1, we have

$$\{\mu_n\}\log\lfloor\mu_n\rfloor = \{\mu_n\}\log k = \log\left(\frac{n}{N_k}\right) + O\left(\frac{1}{k}\right)$$

and

$$\log\left(\frac{N_k + y_nN_k}{N_k}\right) + O\left(\frac{1}{k}\right) = \log(1 + y_n) + O\left(\frac{1}{k}\right).$$

Hence, for $n \in S$,

$$\begin{aligned}\operatorname{Prob}(D_n = \lfloor\mu_n\rfloor) &= \exp(-e^{\log\frac{(1+y_n)}{e\sqrt{2\pi}} + o(1)}) + o(1)\\ &= \exp\left(-\frac{1 + y_n}{e\sqrt{2\pi}}\right) + o(1).\end{aligned}$$

Similarly,

$$\mathrm{Prob}(D_n = \lfloor \mu_n \rfloor + 1) = 1 - \exp\left(-\frac{1+y_n}{e\sqrt{2\pi}}\right) + o(1)\,,$$

and

$$\mathrm{Prob}(D_n = \lfloor \mu_n \rfloor + 2) = o(1)\,. \qquad \blacksquare$$

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APPENDIX 1

This appendix contains a proof of Theorem 1. Let $T(n, k)$ be the number of trees with n labelled vertices and maximum degree less than or equal to k, and let $S_m(t) = \sum_{j=0}^{m} \frac{t^j}{j!}$ denote the mth partial sum of e^t. Moon showed [9] that $\frac{T(n,k)}{(n-2)!}$ is the coefficient of t^{n-2} in $S_{k-1}(t)^n$. It is well known [9] that the total number of trees on n vertices is n^{n-2}. Hence, by Cauchy's theorem,

$$\mathrm{Prob}(D_n \le k) = \frac{(n-2)!}{n^{n-2}(2\pi i)} \oint S_{k-1}(t)^n t^{-n+1}\, dt\,. \tag{9}$$

The integral in (9) will be estimated using the following approximations for partial sums of the exponential series [11, 12]:

Theorem 4. (Szegö) *Let K be any compact subset of the open unit disc. Then, as $m \to \infty$,*

$$S_m(mw) = e^{mw}\left(1 - \frac{1}{\sqrt{2\pi m}}\left(\frac{w}{1-w}\right)(we^{1-w})^m\left(1 + O\left(\frac{1}{m}\right)\right)\right),$$

where the constant implicit in the big O is uniform for $w \in K$.

Let $\langle k_n \rangle_{n=1}^{\infty}$ be any sequence of positive integers for which $k_n \sim (\log n / \log\log n)$ as $n \to \infty$. Then by (9),

$$\mathrm{Prob}(D_n \le k_n) = \frac{(n-2)!}{n^{n-2}(2\pi i)} \oint e^{nH_n(t)} t\, dt\,, \tag{10}$$

where $H_n(t) = \log S_{k_n - 1}(t) - \log(t)$. We are about to use the saddlepoint method to evaluate the integral on the right side of (10).

Toward this end, we first show that there is a unique positive $\rho = \rho_n$ that satisfies the saddlepoint condition. Observe that

$$H_n'(x) = \frac{P_{k_n}(x) - 1}{xS_{k_n-1}(x)} ,$$

where $P_{k_n}(x) := \sum_{j=1}^{k_n-1} \left(\frac{1}{(j-1)!} - \frac{1}{j!} \right) x^j$. Note also that the polynomial $P_{k_n}(x)$ has nonnegative coefficients, and therefore is an increasing function of x. Since $P_{k_n}(0) = 0$, it follows that there is a unique positive solution $x = \rho_n$ to the equation $P_{k_n}(x) = 1$.

Using the saddlepoint method (see Appendix 2), we get

$$\text{Prob}(D_n \leq k_n) = \frac{(n-2)!}{n^{n-2}} \rho_n \frac{e^{nH_n(\rho_n)}}{\sqrt{2\pi n H_n''(\rho_n)}} (1 + o(1)) \tag{11}$$

as $n \to \infty$. In order to make (3) more explicit, we clearly need to estimate ρ_n. The following lemma actually has more precision than we need.

Lemma 2.

$$\rho_n = 1 + \frac{1}{e} \sqrt{\frac{k_n}{2\pi}} \left(\frac{e}{k_n} \right)^{k_n} (1 + o(1)) .$$

Proof. Let $e_n = \rho_n - 1$. It is easy to see that $e_n > 0$ decreases to zero as $n \to \infty$. We want the more precise estimate $e_n = \frac{1}{e} \sqrt{\frac{k_n}{2\pi}} \left(\frac{e}{k_n} \right)^{k_n} (1 + o(1))$. Applying Theorem 4 to the identity

$$\rho_n S_{k_n-2}(\rho_n) - S_{k_n-1}(\rho_n) = 0 , \tag{12}$$

we get

$$\begin{aligned} &(1 + e_n)\left(e^{\rho_n} - \frac{1}{\sqrt{2\pi(k_n-2)}} \left(\frac{\rho_n}{k_n - 2 - \rho_n} \right) \left(\frac{\rho_n e}{k_n - 2} \right)^{k_n-2} \left(1 + O\left(\frac{1}{k_n}\right)\right)\right) \\ &- \left(e^{\rho_n} - \frac{1}{\sqrt{2\pi(k_n-1)}} \left(\frac{\rho_n}{k_n - 1 - \rho_n} \right) \left(\frac{\rho_n e}{k_n - 1} \right)^{k_n-1} \left(1 + O\left(\frac{1}{k_n}\right)\right)\right) = 0 . \end{aligned}$$

Hence

$$e_n = \frac{A_n - B_n}{C_n} , \tag{13}$$

where

$$A_n := \frac{1}{\sqrt{2\pi(k_n-2)}} \left(\frac{\rho_n}{k_n - 2 - \rho_n} \right) \left(\frac{\rho_n e}{k_n - 2} \right)^{k_n-2} \left(1 + O\left(\frac{1}{k_n}\right)\right)$$

$$B_n = \frac{1}{\sqrt{2\pi(k_n-1)}} \left(\frac{\rho_n}{k_n - 1 - \rho_n} \right) \left(\frac{\rho_n e}{k_n - 1} \right)^{k_n-1} \left(1 + O\left(\frac{1}{k_n}\right)\right)$$

and

$$C_n = \left(e^{\rho_n} - \frac{1}{\sqrt{2\pi(k_n-2)}}\left(\frac{\rho_n}{k_n-2-\rho_n}\right)\left(\frac{\rho_n e}{k_n-2}\right)^{k_n-2}\left(1+O\left(\frac{1}{k_n}\right)\right)\right).$$

To simplify the right side of (13), observe that

$$\left(\frac{\rho_n e}{k_n-2}\right)^{k_n-2} = \exp[-k_n \log k_n + k_n + 2\log k_n + o(k_n)]\,. \tag{14}$$

(Here we have used the fact that $\rho_n = 1 + o(1)$.) Since $k_n \sim \dfrac{\log n}{\log\log n}$, we get

$$\left(\frac{\rho_n e}{k_n-2}\right)^{k_n-2} = \exp[-\log n + o(\log n)\} = O(n^{-9})\,.$$

Similarly

$$\left(\frac{\rho_n e}{k_n-1}\right)^{k_n-1} = O(n^{-.9})\,.$$

Putting these back into (13), we get $e_n = O(n^{-9})$. Now we can bootstrap:

$$\begin{aligned}\left(\frac{\rho_n e}{k_n-2}\right)^{k_n-2} &= \exp[-k_n\log k_n + 2\log k_n + 2 + o(1) + k_n - 2 + o(1)]\\ &= \left(\frac{e}{k_n}\right)^{k_n} k_n^2(1+o(1))\,.\end{aligned} \tag{15}$$

Likewise,

$$\left(\frac{\rho_n e}{k_n-1}\right)^{k_n-1} = \left(\frac{e}{k_n}\right)^{k_n} k_n(1+o(1))\,.$$

Finally, putting both of these into (13), we get

$$e_n = \frac{1}{e}\sqrt{\frac{k_n}{2\pi}}\left(\frac{e}{k_n}\right)^{k_n}(1+o(1))\,. \qquad \blacksquare$$

With the lemma at our disposal, each factor in the right side of (11) can be made explicit. First, by Stirling's formula,

$$\frac{(n-2)!}{n^{n-2}} \sim \sqrt{n}e^{-n}\sqrt{2\pi}\,. \tag{16}$$

Second, we have

$$H_n''(\rho_n) = \frac{S_{k_n-1}(\rho_n)S_{k_n-3}(\rho_n) - S_{k_n-2}^2(\rho_n)}{S_{k_n-1}^2(\rho_n)} + \frac{1}{\rho_n^2}\,.$$

Using (12), we see that this is

$$= \frac{S_{k_n-3}(\rho_n)}{S_{k_n-1}(\rho_n)} = \frac{e + o(1)}{e + o(1)} = 1 + o(1)$$

Thus

$$H_n''(\rho_n) = 1 + o(1) . \tag{17}$$

Finally, we have

$$nH_n(\rho_n) = n(\log S_{k_n-1}(\rho_n) - \log \rho_n) .$$

Using Theorem 4, we get

$$\log S_{k_n-1}(\rho_n)$$
$$= \rho_n + \log\left(1 - \frac{1}{\sqrt{2\pi(k_n-1)}}\left(\frac{\rho_n}{k_n-1-\rho_n}\right)\left(\frac{\rho_n e}{k_n-1}\right)^{k_n-1} e^{-\rho_n}\left(1 + O\left(\frac{1}{k_n}\right)\right)\right) .$$

Using the fact that $\log(1-x) = -x + O(x^2)$ as $x \to 0$, we get

$$nH_n(\rho_n) = n - \exp\left[\log n + \log\left(\frac{\rho_n}{\sqrt{k_n-1}(k_n-1-\rho_n)}\right)\right.$$
$$\left. - \log\sqrt{2\pi} + (k_n-1)\log(\rho_n e) - (k_n-1)\log(k_n-1) - \rho_n + O\left(\frac{1}{k_n}\right)\right] + O(n^{-.8}) . \tag{18}$$

Observe that

$$\log\left(\frac{\rho_n}{\sqrt{k_n-1}(k_n-1-\rho_n)}\right) = -\frac{3}{2}\log k_n + o(1) . \tag{19}$$

Combining (16), (17), (18), and (19), we get Theorem 1. ■

APPENDIX 2

Let $\xi := \frac{1}{n^{5/12}}$, and recall that $\rho = \rho_n = 1 + e_n$ where e_n is small. We have

$$\frac{1}{2\pi i}\int_{|z|=\rho} e^{nHn(z)} z\,dz = A + E ,$$

where

$$A = \frac{1}{2\pi}\int_{-\xi}^{\xi} e^{nHn(\rho e^{i\theta})}\rho^2 e^{2i\theta}\, d\theta ,$$

and

$$E = \frac{1}{2\pi} \int_{\xi \le |\theta| \le \pi} S^n_{k_n-1}(\rho e^{i\theta})(\rho e^{i\theta})^{-n+2} \, d\theta \, .$$

For $|\theta| \le \xi$, we have

$$H_n(\rho e^{i\theta}) = H_n(\rho) - \frac{H''_n(\rho)\rho^2\theta^2}{2} + R(n) \, , \tag{20}$$

where

$$R(n) = \frac{\rho^2 H''_n(\rho)}{2} (\theta^2 + (1 - e^{i\theta})^2) + \frac{(\rho e^{i\theta} - \rho)^3}{2\pi i} \int_{|z-\rho|=1/2} \frac{H_n(z)\, dz}{(z-\rho)^3 (z - \rho e^{i\theta})} \, .$$

Note that $H_n(z)$ is uniformly bounded on $|z - \rho| = \frac{1}{2}$. Coupling this with (17), we see that

$$R(n) = O(\xi^3) \, . \tag{21}$$

Using (20), (21), and the method of Laplace, we get

$$\begin{aligned} A &= (1 + o(1)) \frac{e^{nH_n(\rho)}}{2\pi} \int_{-\xi}^{\xi} \exp\left[-\frac{nH''_n(\rho)\theta^2}{2} \right] d\theta \\ &= \frac{e^{nH_n(\rho)}}{\sqrt{2\pi n H''_n(\rho)}} (1 + o(1)) \, . \end{aligned}$$

We still need to check that $E = o(A)$.

$$\begin{aligned} |E| &\le \frac{1}{2\pi} \int_{\xi \le |\theta| \le \pi} |S^n_{k_n-1}(\rho e^{i\theta})| \rho^{-n+2} \, d\theta \\ &\le \rho^{-n+2} S^n_{k_n-1}(\rho) \sup_{\xi \le |\theta| \le \pi} \left| \frac{S^n_{k_n-1}(\rho e^{i\theta})}{S^n_{k_n-1}(\rho)} \right| \, . \end{aligned}$$

Let

$$\Delta_n(\theta) := \frac{1}{\sqrt{2\pi(k_n - 1)}} \, \frac{\rho e^{i\theta}}{k_n - 1 - \rho e^{i\theta}} \left(\frac{\rho e^{i\theta} e}{k_n - 1} \right)^{k_n - 1} e^{-\rho e^{i\theta}} \, .$$

By Theorem 4,

$$\left| \frac{S^n_{k_n-1}(\rho e^{i\theta})}{S^n_{k_n-1}(\rho)} \right| = \left| \frac{e^{n\rho e^{i\theta}}}{e^{n\rho}} \right| \cdot \left| \frac{e^{-n(\Delta_n(\theta) + O(\Delta_n(\theta)/k) + O(\Delta_n(\theta)^2)}}{e^{-n(\Delta_n(0) + O(\Delta_n(0)/k) + O(\Delta_n(0)^2)}} \right| \, .$$

It is not difficult to verify that $\Delta_n(\theta) = e^{-\log n + o(\log n)}$ uniformly with respect to θ. Hence

$$\left|\frac{S^n_{k_n-1}(\rho e^{i\theta})}{S^n_{k_n-1}(\rho)}\right| = e^{n\rho(\cos\theta-1)}\exp[e^{o(\log n)}] < \exp[-n\xi^2/3 + e^{o(\log n)}] < e^{-\frac{1}{6}n^{1/6}}$$

for all sufficiently large n.

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