

PARTITIONS WHOSE PARTS ARE PAIRWISE RELATIVELY PRIME

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Let a_n be the number of (unordered) partitions of n into parts that are “pairwise” relatively prime. In other words, count $\lambda = \{\lambda_1, \lambda_2, \dots\}$ if and only if $\gcd(\lambda_i, \lambda_j) = 1$ for all $i \neq j$. We show that $\log a_n \sim \frac{2\pi}{\sqrt{6}} \sqrt{\frac{n}{\log(n)}}$.

The following problem arose in connection with our research in statistical group theory: estimate $a_n :=$ the number of partitions of n into parts that are pairwise relatively prime. This differs from most problems in the theory of partitions because of the complicated relationship between the part sizes. We obtain an asymptotic formula for $\log a_n$, but leave open the challenging task of obtaining an asymptotic formula for a_n itself.

Let W_n be the set of all partitions of n into parts that are 1's and powers of distinct primes. In the language of generating functions, $\#W_n$ is the coefficient of x^n in

$$\frac{1}{1-x} \prod_{\text{primes } p} (1 + x^p + x^{p^2} + x^{p^3} + \dots).$$

Erdős and Turán [2] showed that

$$\#W_n = \exp\left[\frac{2\pi}{\sqrt{6}} \sqrt{\frac{n}{\log n}} (1 + o(1))\right].$$

Obviously $a_n \geq \#W_n$, and therefore

$$\log(a_n) \geq \frac{2\pi}{\sqrt{6}} \sqrt{\frac{n}{\log n}} (1 + o(1)).$$

It is surprising but true that this inequality is sharp, i.e. we have:

Theorem. $\log a_n \sim \frac{2\pi}{\sqrt{6}} \sqrt{\frac{n}{\log n}}$.

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Proof. It will be convenient to think of partitions as multisets. For example, $\{12, 2^{(3)}, 1^{(2)}\}$ is the partition of 20 with one part of size 12, three parts of size 2, and 2 parts of size 1. Let A_n be the set of all partitions of n whose parts are pairwise relatively prime, and let $a_n = \#A_n$. Now set $v := \left\lfloor \frac{\sqrt{n}}{\log^2 n} \right\rfloor$, and for $\lambda \in A_n$ define the following four multisets:

- $S_1(\lambda) := \{\text{parts of size 1 in } \lambda\}$
- $S_2(\lambda) := \{\text{parts of } \lambda \text{ that are divisible by one or more of the first } v \text{ primes } (2, 3, \dots, q_v)\}$
- $S_3(\lambda) := \{\text{parts of } \lambda \text{ that are powers of primes greater than } q_v\}$
- $S_4(\lambda) := \{\text{remaining parts of } \lambda\} = \{\text{parts of } \lambda \text{ that are products of powers of two or more primes } > q_v\}$

As a specific example, consider the following partition:

$$\begin{aligned} \lambda &= \{7429, 841, 175, 143, 62, 37, 27, 1^{(91286)}\} \\ &= \{(17 \cdot 19 \cdot 23), (29 \cdot 29), (5 \cdot 5 \cdot 7), (11 \cdot 13), (2 \cdot 31), 37, (3 \cdot 3 \cdot 3), 1^{(91286)}\}. \end{aligned}$$

Then $n = 100,000$, $v = 2$, and $q_v = 3$. We therefore have

$$\begin{aligned} S_1(\lambda) &= \{1^{(91286)}\} \\ S_2(\lambda) &= \{62, 27\} = \{(2 \cdot 31), (3 \cdot 3 \cdot 3)\} \\ S_3(\lambda) &= \{841, 37\} = \{(29 \cdot 29), 37\} \\ S_4(\lambda) &= \{7429, 175, 143\} = \{(17 \cdot 19 \cdot 23), (5 \cdot 5 \cdot 7), (11 \cdot 13)\}. \end{aligned}$$

In general, $\lambda \in A_n$ is the (multiset) union of the disjoint multisets $S_i(\lambda)$, $i = 1, \dots, 4$. If we specify $S_i(\lambda)$ for $i = 1, \dots, 4$, then λ is uniquely determined. For $i = 1, \dots, 4$, let $N_i := \#\{S_i(\lambda) \mid \lambda \in A_n\} =$ “the number of ways of choosing $S_i(\lambda)$ ”. Then clearly $a_n \leq \prod_{i=1}^4 N_i$. We shall estimate each of these four factors separately.

(1) Clearly $N_1 \leq n$.

(2) Consider the following sequence of v questions: “Which, if any, part of λ is divisible by the i th prime q_i ?”, $i = 1, \dots, v$. In the preceding example, the answers would have been “62” and “27” respectively. There are $1 + \lfloor n/q_i \rfloor$ possible ways to answer the i th question (the 1 corresponds to the answer “none”). Hence

$$N_2 \leq \prod_{i=1}^v \left(1 + \left\lfloor \frac{n}{q_i} \right\rfloor\right) \leq n^v = \exp\left[O\left(\sqrt{\frac{n}{\log n}}\right)\right].$$

(3) There is an obvious injective map from $\{S_3(\lambda) \mid \lambda \in A_n\}$ to W_n : adjoin $(n - \sum_{\omega \in S_3(\lambda)} \omega)$ parts of size 1. Thus

$$N_3 \leq \#W_n = \exp\left[\frac{2\pi}{\sqrt{6}} \sqrt{\frac{n}{\log n}} (1 + o(1))\right].$$

(4) First we show that if $\lambda \in A_n$ (for n sufficiently large), and $\omega \in S_4(\lambda)$, then ω

is a product of two distinct primes. Clearly ω has at least two prime divisors by the definition of $S_4(\lambda)$. On the other hand, any prime divisor of ω is greater than q_v . Since $q_v \sim \sqrt{n}/2 \log n$, we have $q_v > n^{1/3}$, and therefore ω has at most two prime divisors.

Let $\omega_1 > \omega_2 > \dots > \omega_s$ be the elements of $S_4(\lambda)$ for an arbitrary λ . Then, by the preceding argument, there is a sequence $\langle p_i \rangle_{i=1}^{2s}$ of distinct primes, such that $\omega_i = p_{2i-1} p_{2i}$. If we adopt the convention that $p_{2i-1} < p_{2i}$, then the map $\Phi: \langle \omega_i \rangle_{i=1}^s \mapsto \langle p_i \rangle_{i=1}^{2s}$ is injective. Hence, for each s we have;

$$\begin{aligned} & \#\{S_4(\lambda) \mid \lambda \in A_n \text{ and } S_4(\lambda) \text{ has } s \text{ parts}\} \\ & \leq \#\{\langle p_i \rangle_{i=1}^{2s} \mid \text{the } p_i\text{'s are distinct primes } \leq n\}. \end{aligned}$$

There are $\binom{\pi(n)}{2s}$ ways to choose $2s$ primes that are less than or equal to n , and $(2s)!$ ways to order them. Note that

$$\binom{\pi(n)}{2s} (2s)! < n^{2s} (2s)^{2s}.$$

We therefore have

$$N_4 \leq n \max_s (n^{2s} (2s)^{2s}).$$

If we can show that $s = O(n^{1/3})$, then $N_4 = \exp\left[O\left(\sqrt{\frac{n}{\log n}}\right)\right]$. To bound s , note that

$$\begin{aligned} n & \geq \sum_{i=1}^s \omega_i = \sum_{i=1}^s p_{2i-1} p_{2i} \\ & \geq \sum_{i=1}^s q_i q_{2s-i} \geq cs^3 \log^2 s \\ & \Rightarrow s = O(n^{1/3}). \end{aligned}$$

Combining these bounds, we have

$$a_n \leq \prod_{i=1}^4 N_i = \exp\left[\frac{2\pi}{\sqrt{6}} \sqrt{\frac{n}{\log n}} (1 + o(1))\right]. \quad \square$$

References

- [1] G.E. Andrews, *The Theory of Partitions* (Addison-Wesley, Reading Massachusetts, 1976).
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