Inexpensive d-Dimensional Matchings

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ABSTRACT: Suppose that independent $U(0,1)$ weights are assigned to the $\binom{d}{2}n^2$ edges of the complete d-partite graph with n vertices in each of the $d =$ maximal independent sets. Then the expected weight of the minimum-weight perfect d-dimensional matching is at least $\frac{3}{16}n^{1-(2/d)}$. We describe a randomized algorithm that finds a perfect d-dimensional matching whose expected weight is at most $5d^3n^{1-(2/d)} + d^{15}$ for all $d \geq 3$ and $n \geq 1$. © 2001 John Wiley & Sons, Inc. Random Struct. Alg., 00, 1–09, 2001

1. INTRODUCTION

We consider the problem of finding low-weight perfect matchings in the complete d-partite graph. Let $K_{d\times n}$ denote a complete d-partite graph with n vertices in each maximal independent set. For example, one can construct a $K_{d\times n}$ by labeling dn vertices with the numbers $1, 2, \ldots, dn$, and putting an edge from vertex v to vertex w if and only if $\lceil v/n \rceil \neq \lceil w/n \rceil$. We also use the standard notation K_d for a $K_{d\times 1}$, i.e. a graph with d vertices and all $\binom{d}{2}$ edges. Suppose nonnegative weights are assigned to the $\binom{d}{2}n^2$ edges of $K_{d\times n}$. A perfect matching is a set of n K_ds that cover all the vertices of $K_{d\times n}$. The weight of the perfect matching is just the sum of weights of the $\binom{d}{2}n$ edges that constitute the n K_ds. Consider the following computational problem π_d: given edge-weights on the edges of $K_{d\times n}$ (for some n) as input, find a minimum-weight perfect matching.

For any $d > 2$, π_d is an NP-hard computational problem. To verify this, when $d = 3$, restrict to edge-weights having values 0 and 1 only. This restricted problem

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is equivalent to the pairwise-consistent 3DM problem, which is known to be NP-complete (Garey and Johnson [9] p. 221.) For $d > 3$, there is a simple reduction back to the case $d = 3$. (Roughly: we restrict to instances that have edge-weight 1 for all edges outside a particular induced $K_{3\times n}$.)

In this article, we describe a randomized algorithm for finding a perfect matching in $K_{d\times n}$. The perfect matchings that the algorithm finds are not, in general, of minimum weight. On average, however, the optimum matching and the matching that the algorithm finds both have $\Theta(n^{1-(2/d)})$ expected weight (for fixed d as $n \to \infty$). More precisely, we prove that, if independent $U(0, 1)$ weights are assigned to the edges of $K_{d\times n}$, then the expected weight of the minimum-weight perfect d-dimensional matching is at least $(\frac{1}{4} - 2^{-1-\binom{d}{2}})n^{1-(2/d)}$. We then discuss a randomized algorithm that finds a perfect d-dimensional matching whose expected weight is at most $5d^3n^{1-(2/d)} + d^{15}$ for all $d \geq 3$ and all $n \geq 1$.

We note that, for the classical case $d = 2$, there is a long history. Coppersmith and Sorkin [5] contain recent results and an extensive bibliography. The expected cost converges to $\pi^2/6$ as $n \to \infty$ [1, 2]. Although our proof breaks down for $d = 2$, it is worth remarking that, $\Theta(n^{1-(2/d)})$ is indeed $\Theta(1)$ when $d = 2$. Thus the expected cost of the optimal perfect matching is $\Theta(n^{1-(2/d)})$ for any fixed $d \geq 2$.

As far as we know, the d-dimensional matching problem we are considering has not been studied for $d > 2$. Dyer, Frieze, McDiarmid [6] discuss d-dimensional matching and assignment problems, but their problems differ from the one considered here because their costs are assigned independently to the K_ds, rather than to the edges of the underlying $K_{d\times n}$. Our problem also differs substantially from those considered in the preprints [4, 10] that recently appeared on the Los Alamos eprint archive and are conveniently available at front.math.ucdavis.edu.

In the rest of this article, we assume that uniform $U(0, 1)$ costs are assigned independently to the edges of $K_{d\times n}$ where, unless otherwise specified, $d \geq 3$ and $n \geq 1$.

2. THE LOWER BOUND

Let W^* denote the cost of the minimum-weight perfect matching in $K_{d\times n}$. Then for $d \geq 3$, we have

Theorem 1. $E(W^*) \geq (\frac{1}{4} - 2^{-1-\binom{d}{2}})n^{1-(2/d)}$.

Proof. Let N be the number of K_ds in $K_{d\times n}$ with the property that every edge of the K_d has cost $\frac{1}{2n^{2/d}}$ or less. Then $E(N) = n^d(1/2n^{2/d})^{\binom{d}{2}}$. On the other hand, $E(N) \geq n/2 \Pr(N \geq n/2)$, and therefore

$$\Pr\left(N > \frac{n}{2}\right) \leq \frac{2}{n}n^d\left(\frac{1}{2n^{2/d}}\right)^{\binom{d}{2}} = 2^{1-\binom{d}{2}}.$$

Now let B be the number of K_ds in the optimal perfect matching having cost $1/2n^{2/d}$ or more. Certainly $B \geq n - N$, so

$$\Pr\left(B \geq \frac{n}{2}\right) \geq \Pr(n - N \geq n/2) = 1 - \Pr(N > n/2) \geq 1 - 2^{1-\binom{d}{2}}.$$

Note $W^* \geq B(1/2n^{2/d})$, and consequently $E(W^*|B \geq (n/2)) \geq (n/2)(1/2n^{2/d})$. However, then

$$E(W^*) \geq \Pr\left(B \geq \frac{n}{2}\right)E\left(W^* \mid B \geq \frac{n}{2}\right)$$
$$\geq \left(1 - 2^{1-\binom{d}{2}}\right)\frac{n}{2}\frac{1}{2n^{2/d}} \qquad \blacksquare$$

Corollary 2. *For all* $n \geq 1$ *and all* $d \geq 3$, $E(W^*) > \frac{3}{16}n^{1-(2/d)}$.

3. THE RANDOMIZED APPROXIMATION ALGORITHM

We now describe an algorithm that constructs a perfect matching $\mathcal{M}$ of small cost. It is similar to the algorithm of Karp–Sipser [3, 7] for the case $d = 2$. The perfect matching is grown in stages: we start with $\mathcal{M}_0 = \emptyset$, and add K_ds in the rounds by a randomized procedure described below. This yields partial matchings $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \cdots \subseteq \mathcal{M}$. In each successive round, more expensive K_ds are considered for inclusion in $\mathcal{M}$. Once a K_d Δ is added, it is never removed. Hence, in subsequent rounds, we can rule out all other K_ds that share a vertex with Δ. For any graph H, let $T_{\text{sol}}(H)$ consist of those "solitary" K_ds that meet no other K_d of H. Such K_ds need not be isolated components of H: there can be other incident edges so long as those edges are not part of another K_d. Let H_1 be the subgraph of $K_{d\times n}$ that is obtained by deleting all edges of cost greater than $\xi_1 = c_d n^{-2/d}$, where $c_d = (\sqrt{5}d)^{-(2/d)(d-1)}$. With this notation, Round 1 of the algorithm is very easy to describe:

Round 1: All K_ds in $T_{\text{sol}}(H_1)$ are added to $\mathcal{M}_0$ to form $\mathcal{M}_1$.

Once we have defined H_k for $k > 1$, it will be equally easy to describe round k, namely

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Round k: All K_ds in $T_{\text{sol}}(H_k)$ are added to $\mathcal{M}_{k-1}$ to form $\mathcal{M}_k$.

It remains to describe how H_k is constructed before each iteration $k > 1$. At the beginning of round k, the matching consists of $|\mathcal{M}_{k-1}|$ K_d's. Let $U_k = n - |\mathcal{M}_{k-1}|$ be the number of the number of K_ds that still need to be added to $\mathcal{M}_{k-1}$ to obtain a perfect matching. Thus $U_1 = n$, and $0 \leq U_k \leq U_{k-1}$ for all $k > 1$. Note that if $U_k = m < n$, then, at the beginning of round k, the vertices not covered by $\mathcal{M}_{k-1}$ induce a $K_{d\times m}$. We will construct H_k so that it is a subgraph of this $K_{d\times m}$. Let $\xi_0 = 0$. For $k > 0$ and $U_k \geq 1$, define $\xi_k = \xi_{k-1} + (c_d(1 - \xi_{k-1})/U_k^{2/d})$. (We already defined $\xi_1 = c_d n^{-2/d}$, which is consistent with this definition.) Note that $0 < \xi_k < 1$ and the numbers ξ_k are increasing. These numbers will be used to control the rate of increase in the cost of the K_ds that are considered for inclusion in $\mathcal{M}$.

To remove dependence and simplify our calculations, we define a modified cost matrix C_k as follows: if $k > 1$ and $U_k \geq 1$, then

$$C_k(i, j) = \begin{cases} C(i, j) & \text{if } C(i, j) > \xi_{k-1}, \\ X_k(i, j) & \text{else,} \end{cases}$$

where the $X_k(i,j)$ are i.i.d. and uniform on $(\xi_{k-1}, 1]$. It is important to note that $C(i,j) \le C_k(i,j)$ always. Now let H_k be the subgraph of the $K_{d\times n}$ consisting of all vertices that are not covered by $\mathcal{M}_{k-1}$ at the beginning of round k, and of edges (i,j) between any two of these vertices of cost $C_k(i,j) \le \xi_k$. Note that $H_1, H_2, \ldots, H_{k-1}$ can be determined if one knows the costs of those edges (i,j) with $C(i,j) \le \xi_{k-1}$, as well as the values of the $X_\ell(i,j)$s for $\ell < k$; *no additional information is needed about the costs of edges more expensive than* ξ_{k-1}. The costs $C_k(i,j)$ are uniform on $(\xi_{k-1}, 1]$. Given that $U_k = m < n$, H_k is a random subgraph of a $K_{d\times m}$, with each of the $\binom{d}{2}m^d$ possible edges present in H_k independently with probability

$$p_k = \frac{\xi_k - \xi_{k-1}}{1 - \xi_{k-1}} = c_d m^{-2/d}. \tag{1}$$

This observation was influenced by Karp and Steele [8, p. 199]. We have modified their idea, and iterated it.

Let $L = \lceil 5d \ln n \rceil$. We complete the matching, if necessary, as follows:

Final round: If $\mathcal{M}_L$ is not perfect, then we complete the matching in an arbitrary way, without regard to the cost.

In Lemma 3, we shall prove that each of the first L rounds kills on average a $\frac{1}{5d}$ fraction of the unmatched vertices. It is therefore not surprising that the choice of $L = \lceil 5d \ln n \rceil$ is large enough (see the proof of Corollary 5). The final round may not be needed, and we are able to prove that it does not contribute significantly to the expected cost of the matching $\mathcal{M}$. The algorithm always produces a perfect matching and, for fixed d, it runs in polynomial time.

4. THE UPPER BOUND

We define $t(H)$ to be the number of K_ds in an arbitrary subgraph H of $K_{d\times n}$ and, for every $i = 0, 1, \ldots, d-1$, we define $t_i(H)$ to be the number of K_ds in H that share exactly i vertices with at least one other K_d in H. (So, for example, $t_0(H) = |T_{\mathrm{sol}}(H)|$.) Then we have

Lemma 3. *If* $m \ge d^{13}$ *and* H *is a random subgraph of* $K_{d\times m}$ *obtained by independently including each edge of* $K_{d\times m}$ *with probability* $p = c_d m^{-2/d}$, *then*

$$E(t_0(H)) \ge \frac{m}{5d}.$$

Proof. A K_d is not solitary if it has some number $i > 0$ of vertices that it shares with another K_d. Hence, we have the crude estimate $t_0(H) \ge t(H) - t_{d-1}(H) - t_{d-2}(H) - \cdots - t_1(H)$ and consequently

$$E(t_0(H)) \ge E(t(H)) - \sum_{i=1}^{d-1} E(t_{d-i}(H)). \tag{2}$$

The first term on the right is easy to calculate

$$E(t(H)) = m^d p^{\binom{d}{2}} = \frac{m}{\sqrt{5}d}. \tag{3}$$

To estimate the remaining terms, note that there are m^d ways to choose a K_d in $K_{d\times m}$, and $\binom{d}{d-i}$ ways to choose $d-i$ of its vertices to share with another K_d. There are $\binom{d}{2} - \binom{d-i}{2} = \binom{i}{2} + i(d-i)$ edges in the second K_d that are not in the first K_d. Hence

$$E(t_{d-i}(H)) = m^d p^{\binom{d}{2}} \binom{d}{d-i} (m-1)^i p^{\binom{i}{2}+i(d-i)}.$$

Using this and (3), we get

$$E(t_{d-i}(H)) \leq E(t(H)) \binom{d}{i} m^i p^{\binom{i}{2}+i(d-i)}, \tag{4}$$

$$= E(t(H)) \binom{d}{i} m^{-i(d-i-1)/d} c_d^{\binom{i}{2}+i(d-i)}. \tag{5}$$

For the case $i = d-1$, the right-hand side of (4) is exactly $\frac{1}{\sqrt{5}} E(t(H))$. Hence

$$E(t_1(H)) \leq \frac{1}{\sqrt{5}} E(t(H)). \tag{6}$$

Now let M be a sufficiently large positive constant. (Later we set $M = 13$, but for clarity we leave it as a parameter now.) For $m \geq d^M$, $m^{-i(d-i-1)/d} \leq d^{-Mi(d-i-1)/d}$. Using this and $\binom{d}{i} \leq d^{\min(i,\,d-i)}$, we get from (5)

$$E(t_{d-i}(H)) \leq E(t(H)) d^{f(i)}, \tag{7}$$

where

$$f(i) = \min(i, d-i) - \frac{Mi(d-i-1)}{d} - \frac{(1+\log_d(\sqrt{5}))i(2d-i-1)}{d(d-1)}.$$

If $1 \leq i \leq d-2$, then by choosing M as sufficiently large, we can make f as small as we like. Observe that

$$f(i) \leq -2.25 \tag{8}$$

if

$$M \geq \frac{2.25d}{i(d-i-1)} + \frac{d\min(i, d-i)}{i(d-i-1)} - \frac{2d-i-1}{(d-1)(d-i-1)}. \tag{9}$$

For $M = 13$, one can easily verify that (9) is satisfied for $i = 1$, $i = d/2$, and $i = d-2$. Since $f(i)$ is convex on the intervals $[1, \frac{d}{2}]$ and $[\frac{d}{2}, d-2]$, $f(i) \leq -2.25$ for all i in $[1, d-2]$. From (7) and (8), we get

$$\sum_{i=1}^{d-2} E(t_{d-i}(H)) \leq \frac{d-2}{d^{2.25}} E(t(H)). \tag{10}$$

Finally, combining (10) and (6) with (2), (3), and the assumption $d \geq 3$, we get

$$E(t_0(H)) \geq \left(1 - \frac{d-2}{d^{2.25}} - \frac{1}{\sqrt{5}}\right)E(t(H)) \geq \frac{1}{\sqrt{5}}E(t(H)) = \frac{1}{\sqrt{5}}\frac{m}{\sqrt{5}d} = \frac{m}{5d}. \quad \blacksquare$$

Remark. The lower bound for $E(T_0(H))$ in Lemma 3 is close to the best possible bound obtainable using our methods. To see this, suppose we tried to obtain the better bound $E(t_0(H)) \geq m/4d$ for large enough m. In our proof, we would need to modify the constant c_d to obtain $E(t(H)) = m/x_1 d$ and $E(t_1(H)) \leq (1/x_1)E(t(H))$ for some $x_1 > 1$, and we would need to show $E(t_0(H)) \geq (1/x_0)E(t(H))$ for some $x_0 > 1$ such that $x_0 x_1 = 4$ and $(1/x_0) + (1/x_1) < 1$. There are no values for x_0 and x_1 that would satisfy all the conditions. Therefore, the better bound $E(t_0(H)) \geq (m/4d)$ is not obtainable for any m and constant c_d—using our approach.

Our upper bound is based on the following essential corollary:

Corollary 4. *For all* $k < L = \lceil 5d \ln n \rceil$,

$$E(U_{k+1}) \leq \max\left(d^{13}, \left(1 - \frac{1}{5d}\right)^k n\right).$$

Proof. Clearly $U_1 \leq n$. We proceed by induction on k. By definition, $U_{k+1} = U_k - t_0(H_k)$. If $U_k \leq d^{13}$, then $U_{k+1} \leq U_k \leq d^{13}$. If $U_k > d^{13}$, then we can apply Lemma 3, with $m = U_k$ and $p = p_k$, to get $E(t_0(H_k)|U_k = m) \geq U_k/5d$. But then

$$\begin{aligned} E(U_{k+1}) &\leq E(U_k) - \frac{E(U_k)}{5d} = \left(1 - \frac{1}{5d}\right)E(U_k) \\ &\leq \left(1 - \frac{1}{5d}\right)\max\left(d^{13}, \left(1 - \frac{1}{5d}\right)^{k-1} n\right) \\ &\leq \max\left(d^{13}, \left(1 - \frac{1}{5d}\right)^k n\right). \end{aligned} \quad \blacksquare$$

If the matching is not perfect after L rounds, then the matching is completed in an arbitrary way. Let S be the total weight of the additional K_ds added in this final $(L+1)$st round. Then we have

Corollary 5. $E(S) \leq \binom{d}{2} d^{13}$.

Proof. Each K_d has cost $\binom{d}{2}$ at most. The number of K_ds added in the final clean-up round is at most U_L. Hence

$$E(S) \leq \binom{d}{2} E(U_L) \leq \binom{d}{2} \max\left(d^{13}, \left(1 - \frac{1}{5d}\right)^L n\right).$$

Using the fact that $(1 - (1/x))^x \leq (1/e)$ for all positive x, we get

$$\left(1 - \frac{1}{5d}\right)^L = \left(1 - \frac{1}{5d}\right)^{5d(L/5d)} \leq e^{-(L/5d)} \leq n^{-1}$$

and consequently $E(S) \leq \binom{d}{2} d^{13}$. $\blacksquare$

With Corollary 4 at our disposal, it is now easy to prove the upper bound for $E(W^*)$, our main result:

Theorem 6. *If* $n \geq 1$ *and* $d \geq 3$, *then the expected weight of the perfect matching that the algorithm produces is at most* $5d^3 n^{1-\frac{2}{d}} + d^{15}$.

Proof. If $n \leq d^{13}$, then trivially the weight of $\mathcal{M}$ is at most $n\binom{d}{2} \leq d^{15}$. Otherwise, $U_1 = n > d^{13}$, and we define L' to be the smaller of L and the largest k for which $U_k > d^{13}$. For $k \leq L'$, the number of K_ds added in round k is $U_k - U_{k+1}$. If a K_d is added to $\mathcal{M}_{k-1}$ in round k, then its cost is at most $\binom{d}{2}\xi_k$. Hence, the cost of K_ds added in rounds $1, 2, \ldots, L'$ is at most $\binom{d}{2}\sum_{k=1}^{L'} \xi_k(U_k - U_{k+1})$. Using respectively, summation by parts, the definition of ξ_k, and the fact that $0 < \xi_k < 1$ and $c_d < 1$, we find that

$$\begin{aligned}\sum_{k=1}^{L'} \xi_k(U_k - U_{k+1}) &= \sum_{k=1}^{L'} U_k(\xi_k - \xi_{k-1}) - \xi_{L'}U_{L'+1} \\ &\leq \sum_{k=1}^{L'} U_k\left(\frac{c_d(1-\xi_{k-1})}{U_k^{2/d}}\right) \leq \sum_{k=1}^{L'} U_k^{1-(2/d)}.\end{aligned}$$

By Jensen's inequality $E(U_k^{1-(2/d)}) \leq (E(U_k))^{1-(2/d)}$. Regardless of whether $L' = L$ or $L' < L$, we know by Corollary 5 that the total expected cost of the K_ds added in and after round L' is at most $\binom{d}{2}d^{13}$. Together with Corollary 4, these observations imply that the expected value of the weight of $\mathcal{M}$ is at most

$$\binom{d}{2}\sum_{k=1}^{L'}\left(1 - \frac{1}{5d}\right)^{k(1-(2/d))} n^{1-(2/d)} + \binom{d}{2}d^{13}.$$

If we let $a = 1 - (1/5d)$ and $c = 1 - (2/d)$, then $\sum_{x=1}^{L'} a^{cx} \leq \int_0^\infty a^{cx}dx = -1/(c\ln a) \leq d/(d-2)5d$. Hence the expected weight of $\mathcal{M}$ for $d \geq 3$ is at most

$$5d\frac{d}{d-2}\binom{d}{2}n^{1-(2/d)} + d^{15} \leq 5d^3 n^{1-(2/d)} + d^{15}. \qquad \blacksquare$$

5. DISCUSSION

We proved upper and lower bounds for $E(W^*)$, the expected weight of the minimum-weight perfect d-dimensional matching in $K_{d\times n}$. These bounds are tight in the sense that they are both $\Theta(n^{1-(2/d)})$ as $n \to \infty$ for fixed d. The "constant factor" still represents a large gap, however, if one is interested in allowing $d = d(n)$ to grow with n.

It is easy to prove that, for fixed n, the optimum cost is $\Theta(d^2)$ as $d \to \infty$. Each vertex v must be incident to $d - 1$ edges of the matching, one edge for each part not containing v of the d-partite $K_{d\times n}$. At best, each one of these edges is the cheapest from v to the corresponding part. The expected value of the minimum of n independent $U(0, 1)$ random variables is $1/(n + 1)$. We have

- dn vertices in $K_{d\times n}$, and
- $d-1$ edges of the matching that are incident to each vertex, and
- a $1/(n+1)$ lower bound on the expected cost of each edge of the matching, but
- we are double counting: each edge of the matching is incident to 2 vertices.

Hence, a lower bound for the expected cost of the optimal matching is

$$(dn)\cdot(d-1)\cdot\left(\frac{1}{n+1}\right)\cdot\frac{1}{2}=\frac{n}{n+1}\binom{d}{2}\geq\frac{1}{2}\binom{d}{2}.$$

The upper bound is trivial: there are $n\binom{d}{2}$ edges in the matching, each of which has cost less than 1 and so the expected cost is at most $n\binom{d}{2}$.

After comparing our large-n and large-d results, some readers might speculate that $E(W^*)$ is $\Theta(d^2n^{1-(2/d)})$ in the strong sense that, there are positive constants α,β such that, for all n and all d,

$$\alpha<\frac{E(W^*)}{d^2n^{1-2/d}}<\beta.$$

Our results are consistent with this conjecture, but provide only weak evidence in support of it. The methods in this article do not seem appropriate for estimating $E(W^*)$ in the case where $d=d(n)\to\infty$ as $n\to\infty$.

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